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UNIQUE OPTIMAL TIME MOMENT IN EXPONENTIALLY-CONVEX-REWARD 1-BULLET PROGRESSIVE SILENT DUEL OF ICT INNOVATION LAUNCH

Vadim V. Romanuke

Vinnytsia Institute of Trade and Economics of State University of Trade and Economics, Vinnytsia, Ukraine

Background. In the modern digital economy, the timing of the introduction of new information and telecommunication technologies becomes critical: launching too early may lead to unprepared infrastructure or immature market demand, while delaying too long risks losing the market advantage to competitors. These processes can be interpreted through the lens of progressive silent duels. Each market participant must decide when to act — that is, when to introduce, announce, or deploy a technology — without knowing whether the competitor has already done so. In particular, when the reward of acting grows exponentially with time — representing, for instance, cumulative technological maturity or increasing value of full deployment — yet the risk of being second remains severe, the decision problem aligns with an exponentially-convex-reward duel.

Objective. The paper aims to determine the set of optimal time moments for an exponentially-convex-reward 1-bullet silent duel. From a practical standpoint, the objective of this research is to determine the optimal moment for initiating or announcing a new ICT solution in a competitive environment under the following conditions: readiness and payoff grow progressively over time, information about competitors' actions is unavailable until after both sides have acted, and only one major strategic action is possible within a given competitive cycle. The goal is to identify a stable and universal decision rule, being an optimal strategy of timing, that maximises expected reward under these uncertainty and competition constraints.

Methods. The finite 1-bullet progressive silent duel is considered, in which each of the two duelists shoots with an exponentially-convex reward. The duel is a symmetric matrix game whose optimal value is 0, and the set of optimal strategies is the same for both duelists, regardless of the duel size and how time is quantised. The duel is silenced because the duelist does not learn about the action of the other duelist until the very end of the duel. The duel time quantisation is such that time progresses by the geometrical progression pattern, according to which every following time moment is the partial sum of the respective geometric series. In this duel, the duelist has a single optimal strategy. It is to shoot always at the third time moment, whichever the number of time moments is. Namely, the unique optimal strategy is to shoot at either the duel end moment in the 3×3 duel, or at the three-quarters of the unit time span in bigger duels.

Results. The theoretical finding is that the unique optimal strategy is to act precisely at the third progressive time moment, which is equivalent to around three quarters of the total planning horizon in larger duels. This result suggests that, irrespective of the granularity of internal planning (how finely time is divided into milestones), and regardless of the scale of competition, there exists a universal "sweet spot" for taking action. In real-world terms, this moment corresponds to a late but not final stage of technological preparation — when the solution has reached sufficient maturity and reliability, when market conditions are becoming favourable but not yet saturated, and when the delay is long enough to exploit exponential improvement effects but short enough to avoid being overtaken by a rival.

Conclusions. The "silent" nature of the duel models the real-world asymmetry of competitive information. For enterprises introducing new ICT systems, the third progressive moment represents a strategically balanced readiness threshold. It minimises the risk of premature launch (insufficient maturity) and the threat of excessive delay (competitor's precedence), producing a dominant timing equilibrium that is independent of specific market size or implementation granularity.

Keywords: *timing of innovation; 1-bullet silent duel; time progression; exponentially-convex reward; matrix game; optimal time moment.*

1. Finite 1-bullet progressive silent duels

In the modern digital economy, the introduction of new information and telecommunication technologies (ICT) frequently unfolds as a competitive and time-sensitive process [1], [2]. Enterprises that develop, distribute, or implement innovations such as cloud services, IoT platforms, or 5G-based applications face rivals with comparable competencies and similar goals [3], [4]. The timing of technological introduction becomes critical: launching too early may lead to unprepared infrastructure or immature market demand, while delaying too long risks losing the market advantage to competitors [1], [5], [6].

These processes can be interpreted through the lens of progressive silent duels [7], [8]. Each market participant must decide when to act — that is, when to introduce, announce, or deploy a technology — without knowing whether the competitor has already done so. The silence of the duel reflects informational asymmetry: firms are often unaware of their rivals' exact readiness or launch schedule until outcomes (market reactions, adoption rates, or partnership formations) become publicly observable [2], [9], [10].

In particular, when the reward of acting grows exponentially with time — representing, for instance, cumulative technological maturity or increasing value of full deployment — yet the risk of being second remains severe, the decision problem aligns with an exponentially-convex-reward duel [11], [12]. Thus, the "progressive silent duel" serves as an analytical metaphor for the dynamics of timing in ICT innovation races, where both sides continuously accumulate capability, but must ultimately commit to a single, irreversible decision point [2], [5], [13], [14].

Finite 1-bullet silent duels are used to model one-decision-making competition between two identical intelligent competitors (duelists) through a quantised time span [13], [15], where the duelist benefits from shooting as late as possible but only by acting (or, speaking metaphorically, shooting its single bullet) first [7], [10], [16], [17]. The duel time span is a set

$$T_N = \{t_q\}_{q=1}^{N} \subset [0;1] \text{ by } t_q < t_{q+1}$$
$$\forall\, q = \overline{1, N-1} \text{ and } t_1 = 0,\ t_N = 1 \text{ for } N \in \mathbb{N} \setminus \{1, 2\} \quad (1)$$

of N successive time moments of possible shooting (acting) [11], [17], [18]. The duel is silenced because the duelist does not learn about the action of the other duelist until the very end of the duel $t_N = 1$ [1], [9]. Apart from ICT, such delays are typical for time-lagged systems like decentralised financial systems, jurisprudence, advertising, and general publicising [16], [19], where the system manager temporarily maintains uncertainty with a purpose of accumulating additional advantage for the system development or evolution.

A finite 1-bullet silent duel is a symmetric matrix game

$$\langle X_N, Y_N, \mathbf{U}_N \rangle = \left\langle \{x_i\}_{i=1}^{N}, \{y_j\}_{j=1}^{N}, \mathbf{U}_N \right\rangle$$
$$\text{by } X_N = Y_N = T_N \quad (2)$$

with a skew-symmetric payoff matrix

$$\mathbf{U}_N = [u_{ij}]_{N\times N} = [-u_{ji}]_{N\times N} = -\mathbf{U}_N^{\mathrm{T}} \quad (3)$$

of the first duelist's rewards. The solution of game (2) with (3) and (1) is determined by how moments $\{t_q\}_{q=2}^{N-1}$ are assigned within interval $(0;1)$ and how matrix (3) is structured. Owing to the game symmetry, the set of optimal strategies is the same for both duelists, and the optimal value of the game is 0. Usually, the system manager benefits from that this set contains pure strategies, i.e. optimal time moments to act. Furthermore, the perfect case is when just a single optimal time moment exists. In this way, the system manager latently forces both duelists to act at such a moment. This serves for the system stability and controllability.

Assigning internal moments $\{t_q\}_{q=2}^{N-1}$ of possible shooting must regard the growing tension, responsibility, and plausible anxiety as the duel progresses. Therefore, as the duelist approaches the end moment $t_N = 1$, the space between consecutive moments t_q and t_{q+1}, $q = \overline{1, N-1}$, should not shorten:

$$t_q - t_{q-1} \geqslant t_{q+1} - t_q$$
$$\forall\, q = \overline{2, N-1} \text{ for } N \in \mathbb{N} \setminus \{1, 2\}. \quad (4)$$

In the 3×3 duel set $T_3 = \left\{0, \frac{1}{2}, 1\right\}$ and thus the single internal moment is equally distant from the duel's beginning and end, still obeying (4). In bigger duels, the density of the duelist's pure strategies must gradually grow as the duelist approaches the duel end [7], [8], [14], [20]. One of the patterns of such growth is the geometrical progression, according to which every following moment is the partial sum of the respective geometric series:

$$t_q = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1}-1}{2^{q-1}} \text{ for } q = \overline{2, N-1}.$$

Then game (2) with (3) and

$$T_N = \left\{0, \left\{\frac{2^{q-1}-1}{2^{q-1}}\right\}_{q=2}^{N-1}, 1\right\} \text{ for } N \in \mathbb{N} \setminus \{1, 2\} \quad (5)$$

is a finite 1-bullet progressive silent duel whose time schedule obeys (4) as

$$t_q - t_{q-1} > t_{q+1} - t_q \quad \forall\, q = \overline{2, N-2} \text{ for } N \in \mathbb{N} \setminus \{1, 2, 3\}$$
$$\text{and } t_{N-1} - t_{N-2} = t_N - t_{N-1} = \frac{1}{2^{N-2}} \text{ for } N \in \mathbb{N} \setminus \{1, 2\}. \quad (6)$$

2. Reward exponential rate

In finite 1-bullet silent duels,

$$u_{ij} = g(x_i) - g(y_j) + g(x_i)g(y_j)\operatorname{sign}(y_j - x_i)$$
$$\text{for } i = \overline{1, N} \text{ and } j = \overline{1, N} \quad (7)$$

by some discrete reward functions $g(x_i)$ and $g(y_j)$ of the first and second duelists, respectively, where

$$g(t_1) = g(0) = 0 \text{ and } g(t_N) = g(1) = 1 . \quad (8)$$

Generally speaking, function $g(t_q)$ must be nondecreasing [11], [21], but it is quite appropriate to consider an exponentially-increasing reward function

$$g(t_q) = \frac{e^{t_q} - 1}{e - 1}, \quad (9)$$

which, as it is easy to get convinced, obeys requirements (8). Function (9) is an exponentially-convex-reward function. Upon plugging it into (7), entry u_{ij} of payoff matrix (3) is calculated as

$$u_{ij} = \frac{e^{x_i} - 1}{e - 1} - \frac{e^{y_j} - 1}{e - 1} + \frac{e^{x_i} - 1}{e - 1} \cdot \frac{e^{y_j} - 1}{e - 1} \cdot \operatorname{sign}(y_j - x_i) =$$
$$= \frac{e^{x_i} - e^{y_j}}{e - 1} + \frac{(e^{x_i} - 1)(e^{y_j} - 1)}{(e - 1)^2} \cdot \operatorname{sign}(y_j - x_i)$$
$$\text{for } i = \overline{1, N} \text{ and } j = \overline{1, N} . \quad (10)$$

The objective is to determine the set of optimal time moments $\Theta(N) \subset T_N$ for exponentially-convex-reward 1-bullet silent duel (2) with (5) and (3) as (10). Herein, it ought to be necessarily noted that

$$u_{1j} = \frac{1 - e^{y_j}}{e - 1} + \frac{(1 - 1) \cdot (e^{y_j} - 1)}{(e - 1)^2} = \frac{1 - e^{y_j}}{e - 1} < 0 \quad \forall j = \overline{2, N}$$

and thus the duel beginning moment $t_1 = 0$ is never optimal in such a duel, whichever the number of time moments is:

$$t_1 \notin \Theta(N) \quad \forall N \in \mathbb{N} \setminus \{1, 2\} . \quad (11)$$

From a practical standpoint, the objective of this research is to determine the optimal moment for initiating or announcing a new ICT solution in a competitive environment under the following conditions: readiness and payoff grow progressively over time [3], [22], information about competitors' actions is unavailable until after both sides have acted [23], [24], and only one major strategic action (e. g., product launch, system integration, or partnership announcement) is possible within a given competitive cycle. The goal is to identify a stable and universal decision rule — an optimal strategy of timing — that maximises expected reward under these uncertainty and competition constraints. In applied terms, the model provides guidance for organisations on when to act during the technology development and market preparation process, particularly when the decision is constrained by the exponential growth of potential benefits and silence regarding the competitor's actions.

3. Optimal time moment

Theorem 1. Entry u_{nj} by (10), considered as a discrete function of index $j = \overline{1, n-1}$ by $n \in \{\overline{2, N}\}$, strictly decreases as index j is increased. Entry u_{nj} by (10), considered as a discrete function of index $j = \overline{n+1, N}$ by $n \in \{\overline{2, N-1}\}$, strictly decreases as index j is increased.

Proof. Plugging $i = n$ into (10) for $n \in \{\overline{2, N}\}$, entry

$$u_{nj} = \frac{e^{x_n} - e^{y_j}}{e - 1} - \frac{(e^{x_n} - 1)(e^{y_j} - 1)}{(e - 1)^2} =$$
$$= \frac{e^{x_n} - 1}{e - 1} - \left(1 + \frac{e^{x_n} - 1}{e - 1}\right) \cdot \frac{e^{y_j} - 1}{e - 1}$$
$$\text{for } j = \overline{1, n-1} \text{ at } n \in \{\overline{2, N}\} . \quad (12)$$

Due to $e^{x_n} > 1$ and $e^{y_j} \geqslant 1$ by $x_n > 0$ and $y_j \geqslant 0$, respectively, entry (12) is a negatively-sloped line with respect to exponent e^{y_j}. Therefore, entry (12) strictly decreases as index j is increased off 1 up to $n - 1$.

Plugging $i = n$ into (10) for $n \in \{\overline{2, N-1}\}$, entry

$$u_{nj} = \frac{e^{x_n} - e^{y_j}}{e - 1} + \frac{(e^{x_n} - 1)(e^{y_j} - 1)}{(e - 1)^2} =$$
$$= \frac{e^{x_n} - 1}{e - 1} - \left(1 - \frac{e^{x_n} - 1}{e - 1}\right) \cdot \frac{e^{y_j} - 1}{e - 1}$$
$$\text{for } j = \overline{n+1, N} \text{ at } n \in \{\overline{2, N-1}\} . \quad (13)$$

Inasmuch as

$$1 > \frac{e^{x_n} - 1}{e - 1} > 0 ,$$

entry (13) is a negatively-sloped line with respect to exponent e^{y_j}. Therefore, entry (13) strictly decreases as index j is increased off $n + 1$ up to N. □

Theorem 2. Whichever the number of time moments is in exponentially-convex-reward 1-bullet silent duel (2) with (5) and (3) as (10), the set of optimal time moments is a singleton and invariant with respect to the moment index:

$$\Theta(N) = \{t_3\} \quad \forall N \in \mathbb{N} \setminus \{1, 2\} . \quad (14)$$

Proof. In the 3×3 duel, time moment $t_3=1$ is single optimal if the third row of matrix (3) is positive except for entry $u_{33}=0$. Herein,

$$u_{31}=\frac{e^{x_3}-e^{y_1}}{e-1}-\frac{\left(e^{x_3}-1\right)\left(e^{y_1}-1\right)}{(e-1)^2}=$$
$$=\frac{e^1-e^0}{e-1}-\frac{\left(e^1-1\right)\left(e^0-1\right)}{(e-1)^2}=1$$

and

$$u_{32}=\frac{e^{x_3}-e^{y_2}}{e-1}-\frac{\left(e^{x_3}-1\right)\left(e^{y_2}-1\right)}{(e-1)^2}=$$
$$=\frac{e^1-\sqrt{e}}{e-1}-\frac{\left(e^1-1\right)\left(\sqrt{e}-1\right)}{(e-1)^2}=$$
$$=\frac{e-2\sqrt{e}+1}{e-1}>0.2449,$$

i. e. time moment $t_3=1$ is single optimal in the 3×3 duel.

Time moment $t_3=\frac{3}{4}$ is single optimal in an $N\times N$ duel by $N\in\mathbb{N}\setminus\{1,2,3\}$ if inequalities

$$u_{3j}=\frac{e^{x_3}-e^{y_j}}{e-1}-\frac{\left(e^{x_3}-1\right)\left(e^{y_j}-1\right)}{(e-1)^2}>0$$
$$\forall\, y_j<x_3=\frac{3}{4} \text{ by } j=1,2 \tag{15}$$

and

$$u_{3j}=\frac{e^{x_3}-e^{y_j}}{e-1}+\frac{\left(e^{x_3}-1\right)\left(e^{y_j}-1\right)}{(e-1)^2}>0$$
$$\forall\, y_j>x_3=\frac{3}{4} \text{ by } j=\overline{4,N} \tag{16}$$

hold. Owing to Theorem 1, function u_{2j} by (15) is decreasing with respect to index $j=1,2$, and hence inequality (15) is equivalent to inequality:

$$u_{32}=\frac{e^{x_3}-e^{y_2}}{e-1}-\frac{\left(e^{x_3}-1\right)\left(e^{y_2}-1\right)}{(e-1)^2}=$$
$$=\frac{e^{x_3}e-e^{x_3}-e^{y_2}e+e^{y_2}-e^{x_3}e^{y_2}+e^{y_2}+e^{x_3}-1}{(e-1)^2}=$$
$$=\frac{e^{x_3}e-e^{y_2}e+2e^{y_2}-e^{x_3}e^{y_2}-1}{(e-1)^2}=$$
$$=\frac{e^{\frac{3}{4}}e-e^{\frac{1}{2}}e+2e^{\frac{1}{2}}-e^{\frac{3}{4}}e^{\frac{1}{2}}-1}{(e-1)^2}=$$
$$=\frac{e^{\frac{7}{4}}-e^{\frac{3}{2}}+2e^{\frac{1}{2}}-e^{\frac{5}{4}}-1}{(e-1)^2}>0.0271,$$

i. e. inequalities (15) hold.

Owing to Theorem 1, function u_{3j} by (16) is decreasing with respect to index $j=\overline{4,N}$, and hence inequality (16) is equivalent to inequality

$$u_{3N}=\frac{e^{x_3}-e^{y_N}}{e-1}+\frac{\left(e^{x_3}-1\right)\left(e^{y_N}-1\right)}{(e-1)^2}=$$
$$=\frac{e^{x_3}-e}{e-1}+\frac{\left(e^{x_3}-1\right)(e-1)}{(e-1)^2}=\frac{2e^{x_3}-e-1}{e-1}=$$
$$=\frac{2e^{\frac{3}{4}}-e-1}{e-1}>0.3001,$$

i. e. inequalities (16) hold. □

4. Limitation of time uniform quantisation

It is worth noting that just the time progression by (5) creates the duel solution invariant by (14), when the exponential growth factor is 1 by (9). Thus, when

$$T_N=\left\{t_q\right\}_{q=1}^{N}=\left\{\frac{q-1}{N-1}\right\}_{q=1}^{N}\subset[0;1]$$
$$\text{for } N\in\mathbb{N}\setminus\{1,2\}, \tag{17}$$

no invariant exists for exponentially-convex-reward 1-bullet silent duel (2) with (17) and (3) as (10). In particular, if the time progression by (5) is substituted with the time uniform quantisation by (17), then pure strategy solutions exist only when the duelist has three to five time moments to shoot [23]:

$$\Theta(3)=\{t_3\}=\{1\},$$
$$\Theta(4)=\{t_3\}=\left\{\frac{2}{3}\right\},$$
$$\Theta(5)=\{t_4\}=\left\{\frac{3}{4}\right\},$$

whereas

$$\Theta(N)=\varnothing\quad\forall\, N\in\mathbb{N}\setminus\left\{\overline{1,5}\right\}.$$

If time progresses by set (17) and, instead of exponentially-convex-reward function (9) for (7), linear-reward function $g\left(t_q\right)=t_q$ is used [13], set

$\Theta(N)$ bears some resemblance to that of the exponentially-convex-reward duel with the time uniform quantisation by (17), but the system manager must not schedule the duel along eight time moments or more:

$$\Theta(3)=\{t_2, t_3\}=\left\{\frac{1}{2}, 1\right\},\ \Theta(4)=\{t_3\}=\left\{\frac{2}{3}\right\},$$

$$\Theta(5)=\{t_3\}=\left\{\frac{1}{2}\right\},\ \Theta(7)=\{t_4\}=\left\{\frac{1}{2}\right\},$$

whereas

$$\Theta(6)=\varnothing \text{ and } \Theta(N)=\varnothing \quad \forall N \in \mathbb{N} \setminus \left\{\overline{1,7}\right\}.$$

So, the time progression by (5) along with the naturally nonlinear reward growth by (9) ensure the stability and controllability of discrete systems modelled by silent duels [2], [7], [15], [16].

5. Discussion

The 1-bullet model imposes a fundamental limitation because it allows for only one strategic, irreversible action by each player. In real ICT markets, competitive interaction is typically multi-stage and iterative, involving sequential phases of development, testing, pilot deployment, scaling, and subsequent updates of technological solutions. Under such conditions, decisions made at early stages may be revised or adjusted later.

Thus, the 1-bullet duel does not account for adaptive learning, market feedback, or reactive strategy adjustments after observing competitors' actions. It also abstracts from reputational effects, dynamic investment decisions, and the gradual accumulation of competitive advantages. For this reason, the obtained result should not be interpreted as a direct prescription for the entire innovation cycle, but rather as a characterisation of a single, critical decision moment.

Despite these limitations, the result of the 1-bullet model can be interpreted as representative of the early phase of a multi-shot innovation cycle, when future actions are still undefined, and the key decision concerns the timing of the first substantial market entry. At this stage, competition often exhibits the characteristics of a silent duel: information about competitors' readiness is limited, and the first significant action has a disproportionately large impact on the subsequent development trajectory.

In this context, the 1-bullet model may be viewed as a local approximation of a more complex dynamic game, capturing the initial "decision node" of a multi-stage process. The resulting unique optimal timing strategy should then be interpreted not as the final launch time of the entire technology, but as the optimal moment for the first irreversible commitment (such as a public announcement, a large-scale pilot, or entry into a strategic partnership).

Thus, the 1-bullet progressive silent duel does not aim to fully describe multi-shot innovation competition, but provides an analytically rigorous and interpretable result for the critical initial stage of strategic timing decisions. A multi-stage innovation process can be viewed as a sequence of critical decision moments, each of which requires an agent to either undertake or postpone a single, irreversible action. Although the overall interaction is repeated, each stage locally exhibits the structure of a 1-bullet game: the player either "shoots" (commits) or continues to wait.

In this sense, the 1-bullet progressive silent duel serves as an elementary building block of a more complex dynamic game. The multi-stage model emerges through recursive repetition of such local duels, where each subsequent subgame starts under new initial conditions determined by the outcome of the previous stage.

Formally, after the completion of each stage of the innovation cycle, a subgame arises in which the level of technological maturity is updated, the reward function is modified (scaled or shifted), the time horizon is shortened or renormalised, and the information structure again becomes silent with respect to future competitor actions. Under fixed subgame conditions, the strategic choice reduces to determining a single optimal timing decision. Therefore, each subgame in isolation constitutes a 1-bullet progressive silent duel with exponentially convex reward, for which a unique optimal strategy continues to exist.

If the reward structure preserves its exponentially convex form at each stage, the optimal strategy in each subgame can be determined independently of future stages. This implies that the strategy is dynamically consistent: the optimal choice in the current subgame remains optimal regardless of which subsequent subgames may arise. In this case, the multi-stage game admits a solution via backward induction, where the decision at each stage reduces to solving the corresponding 1-bullet duel. Consequently, the unique optimal third progressive timing moment constitutes a locally stable decision rule at each stage of the innovation process.

In practical ICT applications, this recursive structure corresponds to staged technology deployment: each phase (announcement, pilot, scaling, upgrade) creates a new cycle of waiting and decision-making. Although competition is multi-shot overall, each cycle involves a local problem of choosing the timing of an irreversible action under limited information. Thus, the 1-bullet model does not contradict the multi-stage reality; rather, it provides a micro-level decision logic that is recursively embedded within the broader dynamics of innovation competition.

A multi-stage innovation competition can be interpreted as a recursive sequence of 1-bullet progressive silent duels, each defined by updated reward conditions and time horizons. As long as the exponentially convex reward structure is preserved, the unique optimal third progressive timing moment remains locally optimal in each subgame, ensuring dynamic consistency of the multi-stage strategy.

6. Conclusion

In exponentially-convex-reward 1-bullet progressive silent duel (2) with (5) and (3) as (10), the duelist has the single optimal strategy. It is to shoot always at the third time moment, whichever the number of time moments is, making thus this solution a unique invariant. Namely, the unique optimal strategy is to shoot at either the duel end moment in the 3×3 duel, or at the three-quarters of the unit time span in bigger duels.

The theoretical finding — that the unique optimal strategy is to act precisely at the third progressive time moment (equivalently, around three quarters of the total planning horizon in larger duels) — translates into a surprisingly stable rule for ICT-related competition. This result suggests that, irrespective of the granularity of internal planning (how finely time is divided into milestones), and regardless of the scale of competition, there exists a universal "sweet spot" for taking action. In real-world terms, this moment corresponds to a late but not final stage of technological preparation — when the solution has reached sufficient maturity and reliability, when market conditions are becoming favourable but not yet saturated, and when the delay is long enough to exploit exponential improvement effects but short enough to avoid being overtaken by a rival. Hence, for enterprises introducing new ICT systems, the third moment represents a strategically balanced readiness threshold. It minimises the risk of premature launch (insufficient maturity) and the threat of excessive delay (competitor's precedence), producing a dominant timing equilibrium that is independent of specific market size or implementation granularity.

The research confirms that in exponentially progressive competitive environments, the decision "when to act" can have a single dominant optimum. For technology innovators, this means that launch timing should not be arbitrary or purely reactive — it should follow a structurally defined rule derived from reward progression patterns.

The fact that the same optimal timing persists regardless of how the time horizon is discretized implies a scale-invariant principle of decision-making. Whether strategic planning spans months or years, the optimal readiness point remains proportionally fixed — approximately three-quarters through the planning cycle.

Acting too early in technological competition usually underutilises the potential exponential payoff of readiness, while acting too late risks being pre-empted. The third progressive moment rule offers an operational balance between these extremes, guiding firms toward efficient deployment timing. This is a risk-reward balance for ICT innovators.

The "silent" nature of the duel models the real-world asymmetry of competitive information. Organisations should therefore treat the absence of reliable intelligence about competitors not as a drawback, but as a structural feature of the decision environment — one that can be strategically managed using timing-based optimisation. For policymakers and ICT managers, these results highlight the importance of integrating mathematically grounded timing models into innovation management frameworks, particularly where unit-factor exponential growth of technological reward is expected.

A key economic insight of the study lies in the contrast between uniform and progressive growth of readiness to act and expected reward. Under uniform growth, each additional unit of waiting increases the attractiveness of delay by approximately the same amount. In such conditions, the strategic timing becomes unstable: small changes in beliefs, information, or expectations about the competitor's actions may shift the optimal moment over a wide range or even destroy its uniqueness. Economically, this corresponds to a situation of "permanent waiting", where no moment is qualitatively superior to its neighbors, and strategic behavior becomes excessively sensitive to noise and speculative signals.

By contrast, progressive growth of readiness and reward (in particular, exponentially convex growth in progressive time) introduces asymmetry between early and late decision moments. Each additional unit of waiting yields an increasing marginal benefit, while simultaneously raising the strategic cost of losing initiative. This creates a well-defined balance between the value of waiting and the cost of delay.

From an economic perspective, a progressive reward structure endogenously disciplines strategic choice: the optimal timing becomes stable, locally isolated, and robust to small perturbations. This explains why progressive growth ensures the controllability of decisions in competitive ICT innovation processes, whereas uniform growth tends to generate strategic instability.

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Романюк В.В.

Єдиний оптимальний момент часу в однокульовій прогресивній безшумній дуелі з експоненціально-опуклою винагородою у запуску ІКТ-інновацій

Вінницький торговельно-економічний інститут Державного торговельно-економічного університету, м. Вінниця, Україна

Проблематика. У сучасній цифровій економіці час упровадження нових інформаційних і телекомунікаційних технологій набуває критичного значення: надто ранній запуск може призвести до неготовності інфраструктури або незрілого ринкового попиту, тоді як надмірне зволікання несе ризик втрати конкурентної переваги. Такі процеси можна інтерпретувати крізь призму прогресивних безшумних дуелей. Кожен учасник ринку має вирішити, коли діяти — тобто коли впроваджувати, оголошувати чи розгортати технологію — не знаючи, чи вже зробив це конкурент. Зокрема, коли виграш від дії зростає експоненціально з часом — відображаючи, наприклад, кумулятивне технологічне дозрівання або збільшення вартості повного розгортання, — але ризик опинитися другим залишається значним, задача прийняття рішення узгоджується з моделлю дуелі з експоненціально-опуклою винагородою.

Мета дослідження. Метою є визначення множини оптимальних моментів часу в однокульовій прогресивній безшумній дуелі з експоненціально-опуклою винагородою. З практичного погляду, мета цього дослідження полягає у визначенні оптимального моменту для ініціювання або оголошення нового рішення у сфері інформаційних та телекомунікаційних технологій (ІКТ) у конкурентному середовищі за таких умов: рівень готовності та потенційна винагорода зростають поступово з часом; інформація про дії конкурентів відсутня до моменту, коли обидві сторони вже здійснили свої кроки; у межах одного конкурентного циклу можлива лише одна ключова стратегічна дія. Завданням є ідентифікація стабільного та універсального правила прийняття рішень, яке було б оптимальною стратегією моменту дії, що максимізує очікувану винагороду в умовах невизначеності та конкуренції.

Методика реалізації. Розглядається скінченна однокульова прогресивна безшумна дуель, у якій кожен із двох дуелянтів робить постріл з експоненціально-опуклою винагородою. Дуель є симетричною матричною грою з нульовим оптимальним значенням, і множина оптимальних стратегій є однаковою для обох дуелянтів, незалежно від розміру дуелі та способу квантування часу. Дуель є безшумною, оскільки один дуелянт не знає про дію іншого дуелянта до самого завершення дуелі. Квантування часу побудовано за геометричною прогресією, згідно з якою кожен наступний момент часу є частковою сумою відповідного геометричного ряду. У цій дуелі дуелянт має лише одну оптимальну стратегію — завжди стріляти в третій момент часу, незалежно від кількості часових моментів. Така єдина оптимальна стратегія полягає в тому, щоб стріляти або в кінцевий момент у 3×3 -дуелі, або на трьох чвертях одиничного часового інтервалу у більших дуелях.

Результати дослідження. Теоретичний результат полягає в тому, що єдиною оптимальною стратегією є дія саме в третьому прогресивному моменті часу, що еквівалентно приблизно трьом чвертям усього планового горизонту в більших дуелях. Цей результат свідчить, що незалежно від деталізації внутрішнього планування (тобто від того, наскільки дрібно поділено час на етапи), а також незалежно від масштабу конкуренції, існує універсальна “точка рівноваги дії”. У реальних умовах цей момент відповідає пізньому, але ще не завершальному етапу технологічної підготовки — коли рішення досягло достатнього рівня зрілості та надійності, коли ринкові умови стають сприятливими, але ще не насиченими, і коли затримка є достатньо довгою, щоб використати експоненціальний ефект покращення, але не настільки великою, щоб суперник встиг обігнати.

Висновки. “Безшумна” природа дуелі моделює реальну асиметрію конкурентної інформації. Для підприємств, що впроваджують нові ІКТ-системи, третій прогресивний момент часу є стратегічно збалансованим порогом готовності. Він мінімізує ризик передчасного запуску (недостатня зрілість технології) та загрозу надмірного зволікання (перевага конкурента), формуючи домінантну рівновагу часу дії, яка не залежить від конкретного розміру ринку чи рівня деталізації впровадження.

Ключові слова: *таймінг інновації; однокульова безшумна дуель; часова прогресія; експоненціально-опукла винагорода; матрична гра; оптимальний момент часу.*

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