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THEORY OF GUIDED WAVES IN THE INFINITE SYSTEMS OF COUPLED DIELECTRIC RESONATORS

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Background. One of the promising elements of optical and quantum communication systems is various delay lines built on the high-quality dielectric resonators (DRs). These lines typically comprise a substantial number of elements, making the optimisation of their parameters quite challenging. The theory of DRs serves as a foundation for comprehending, calculating, and optimising the parameters of delay lines and other devices, facilitating a considerable reduction in the computational resources that typically require the use of powerful computers.

Objective. The study aims to derive analytical expressions for the electromagnetic parameters of diverse optical waveguides, composed of numerous types of DRs, to utilise them as transmission lines for optical communication systems. To address this issue, an infinite linear system of equations has been derived based on the perturbation theory applied to Maxwell's equations, which connects the complex amplitudes, wave numbers and the resonator frequencies.

Methods. To derive solutions for the analytical expressions, perturbation theory and the theory of infinite linear equations are employed. The outcome is a set of new general analytical formulae that describe the dispersion curves of lattices made up of an infinite number of various types of DRs.

Results. A theory of wave propagation in systems of interconnected one-, two-, and three-dimensional lattices of DRs extended infinitely in one or more directions has been developed. New analytical expressions for the dispersion characteristic of eigenwaves, delay times, and distributions of complex amplitudes of resonators, without any limitations on their quantity, have been derived. By utilising perturbation theory, a novel analytical model has been developed that describes the eigenwaves of three-dimensional lattices composed of identical ring structures of DRs. General analytical solutions for frequency dependencies and amplitudes for one-, two-, and three-dimensional lattices with varying arrangements of resonators have been identified.

Conclusions. The developed theory serves as the foundation for the analysis and design of many devices operating within the optical wavelength spectrum, constructed upon an infinite variety of distinct types of DRs. The obtained new analytical expressions for calculating optical waveguide parameters, based on coupled oscillations of DRs, enable the development of innovative and more efficient mathematical models for various optical communication devices.

Keywords: *dielectric resonator; eigen oscillations; lattice; coupled resonator transmission line; waveguide; perturbation theory; delay line.*

I. INTRODUCTION

Different waveguide elements build on the coupled DRs [1 – 38] are applied today in various devices of the communication systems, such as delay lines [22 – 27, 30, 32, 34 - 39]; metalens [1], modulators [2, 4, 6]; lasers [9, 17], filters [13, 29, 31] and so on. Waveguides on the DR may be made on a 1-dimensional [15, 19, 22-27, 30, 32-36, 38], 2-dimensional [2, 3, 5 – 10, 15 – 16, 19, 20, 28, 29, 37], as well as 3-dimensional lattices [1, 4, 11 - 14, 19, 31]. To analyse the electromagnetic characteristics of such complex waveguide structures, the concept of effective permittivity and effective magnetic permeability is introduced (see, for example, [34]). Sometimes, the calculation of parameters of the lattice is performed by using numerical methods, which requires significant computer resources. Meanwhile, all necessary parameters of waveguides can be analysed and calculated in analytical form using perturbation theory [42]. Obtaining analytical expressions for such complex infinite lattices of coupled DRs allows us to significantly simplify their analysis and optimisation.

II. WAVES IN THE INFINITE DR LATTICES

The real part of the difference between the frequencies of adjacent oscillations of a lattice consisting of a N DR decreases proportionally N^{-1}. At the same time, it can be assumed that the highest quality oscillations will have a greater influence. These qualitative considerations explain the possibility of transition to a continuous real spectrum with an unlimited increase in the number of resonators when describing their coupled oscillations.

Using perturbation theory, from a general perspective, we considered the problem of propagation of waves in unlimited systems of coupled DRs, determined by the electric and magnetic field strengths $(\mathbf{e}, \mathbf{h})$, represented in terms of the natural oscillations of the same isolated resonators $(\mathbf{e}_s, \mathbf{h}_s)$.

Following a perturbation method, similar to [42], we expanded the field of coupled oscillations $(\mathbf{e}, \mathbf{h})$ near one of the natural frequency ω_0 of the isolated resonators over the field of their natural oscillations $(\mathbf{e}_s, \mathbf{h}_s)$:

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{h} \end{pmatrix} = \sum_{s=-\infty}^{\infty} b_s \begin{pmatrix} \mathbf{e}_s \\ \mathbf{h}_s \end{pmatrix}; \qquad (1)$$

To calculate the expansion coefficients $\|b_s\|$ of (1), we used the relationships that follow from Maxwell's equations for the natural oscillations of isolated resonators, as well as for coupled resonators. After transformations of the fields, carried out similarly to [42], an infinite system of linear equations for the amplitudes was obtained:

$$\sum_{s=-\infty}^{\infty} \kappa_{st}\, b_s - \lambda b_t = 0; \quad \left(t = -\infty, ..., \infty\right), \qquad (2)$$

where κ_{st} - are coupling coefficients of a s–th and t–th DR; diagonal elements of the infinite matrix determined only by the magnitude of the radiation of s-th partial resonators, represented by the coupling coefficient $\tilde{k}_s$.

One of the differences between system (2) and found in [42] is that the oscillation frequency ω in (2) takes only real values. Given the definition

$$\lambda = 2(\omega - \omega_0) / \omega_0; \qquad (3)$$

we arrive at the requirement that the eigenvalues λ must also take only real values. Here ω_0 - real part of the frequency of isolated DRs.

It was assumed to that the determinant, and in addition, all minors of the matrix:

$$\det(K - \lambda I) = \det \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & i\tilde{k}_{t-1} - \lambda & \kappa_{t(t-1)} & \kappa_{(t+1)(t-1)} & \cdots \\ \cdots & \kappa_{(t-1)t} & i\tilde{k}_t - \lambda & \kappa_{(t+1)t} & \cdots \\ \cdots & \kappa_{(t-1)(t+1)} & \kappa_{t(t+1)} & i\tilde{k}_{t+1} - \lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \qquad (4)$$

converge to a finite limit in the sense [39].

Here $\kappa_{st} \neq \kappa_{ts}$ - are coupling coefficients of a s–th and t–th for different DR.

It seems that under such complex restrictions, the solution of the system of equations (2) does not exist, however, as will be shown below, using the examples of a system of coupled identical resonators, placed at the nodes of an infinite in one or several directions, lattice, at equal distances from each other, the necessary solutions can be constructed in analytical form.

III. WAVE PROCESSES IN ONE-DIMENSIONAL INFINITE LATTICES OF DRs

Initially, an infinite one-dimensional lattice of identical DRs, located at the same distance from each other, was considered. Under such conditions $\kappa_{sn} = \kappa_{|s-n|}$, $\tilde{k}_s = \tilde{k}_0$ and the system of equations (2) takes the form:

$$(i\tilde{k}_0 - \lambda) b_t + \sum_{s \neq t = -\infty}^{\infty} \kappa_{|s-t|}\, b_s = 0; \quad \left(t = -\infty, ..., \infty\right), \qquad (5)$$

We sought the solution of (5) in the form of propagating harmonic waves of the amplitudes of partial resonators [41]:

$$b_n = b_0 e^{\mp i\psi n} \qquad (6)$$

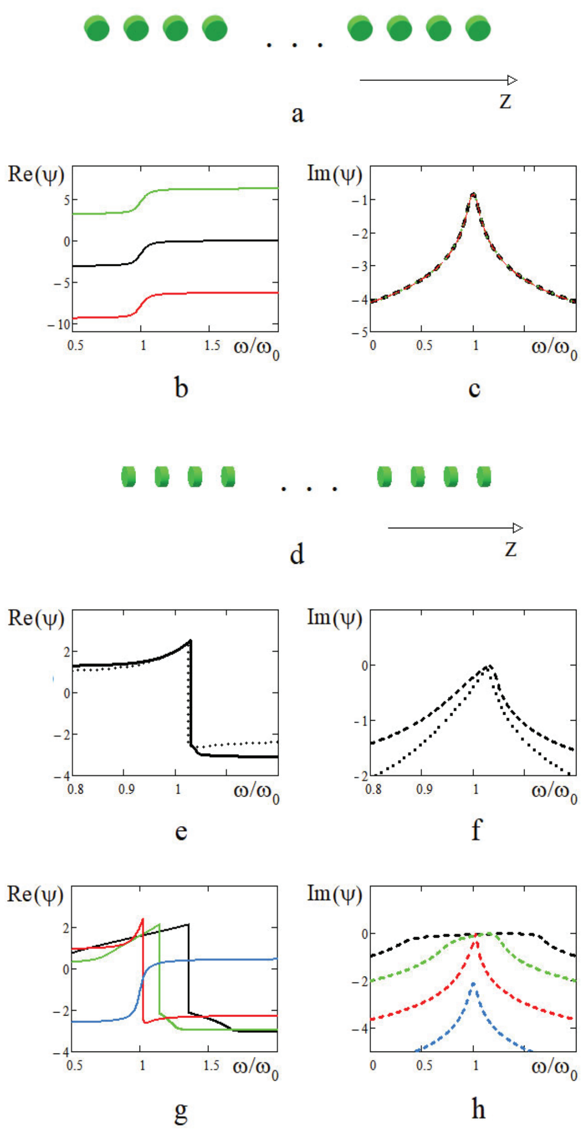

Fig. 1. One-dimensional infinite lattices (a, d) of identical cylindrical DR with H_{101} oscillations. Dependencies of the real (b, e, g) and imaginary (c, f, h) parts of the longitudinal wave parameter on the relative frequency for resonators made of dielectric $\varepsilon_{1r} = 20$; relative sizes: $\Delta = L/2r_0 = 0,4$ for $s = -1;0;1$ (9) (a – c); for $s = 0$; $k_0\Delta z = 0,7;1;2;5$ (g, h) black, green, red, blue lines. The distance between the centers of adjacent resonators: $k_0\Delta z = 2$; $k_0 = \omega_0 / c$; c speed of light.

The complex parameter ψ, defined from Floquet's theorem, is not a wave number in the generally accepted sense, since the representations (1) are not expansions in plane waves. Solutions (6) display the wave process in the system of coupled resonators, expressed by a more complex spatial distribution of fields. In this sense, the real part ψ corresponds to the "wavelength number" γ, multiplied by the distance between the centers of adjacent resonators Δz, and the imaginary part ψ determines the "damping" of the fields of DR at the real oscillation frequency:

$$\psi = \gamma \cdot \Delta z \qquad (7)$$

Substituting (6) into (5) and combining the terms located symmetrically with respect to the t-th resonator, we obtained the characteristic equation:

$$(i\tilde{k}_0 - \lambda) + 2\sum_{s=t+1}^{\infty} \kappa_{|s-t|} \cos(s\psi) = 0\,. \qquad (8)$$

Given (3), we rewrote (8) as:

$$\frac{\omega}{\omega_0} = 1 + \frac{i}{2}\tilde{k}_0 + \sum_{s=1}^{\infty} \kappa_s \cos(s\psi)\,. \qquad (9)$$

The expression obtained is an even function ψ; it defines the same dependence of the resonator amplitudes (6) for waves propagating in opposite directions. In the approximation $s = 1$ an equation, similar to (9), was obtained in [38].

Taking into account the accepted dependence of solutions on time, proportional to $e^{i\omega t}$, equation (9) was supplemented by the condition: $\mathrm{Im}(\psi) \le 0$, corresponding to the requirement of decreasing resonator amplitudes due to radiation energy losses.

In the first approximation, the dependence $\psi(\omega)$ can be simply calculated, taking into account in (9) the coupling between only neighbouring resonators:

$$\psi(\omega) \approx \pm \arccos\left[(2\frac{\omega - \omega_0}{\omega_0} - i\tilde{k}_0)/2\kappa_1 \right] \pm 2s\pi\,. \qquad (10)$$

From where we see, that $\psi(\omega)$ is the periodic function in the $\mathrm{Re}[\psi(\omega)]$ direction defined by $s = 0,1,2,...,\infty$, at that $\mathrm{Im}[\psi(\omega)]$ do not depend on s.

Fig. 1 (b, c; e - h) show the dependencies of the real and imaginary parts ψ on the frequency for one-dimensional lattices of cylindrical DRs, in the region of fundamental oscillations H_{101} of magnetic types, found using the approximate formula (9) and by the exact solution of equation (8) (points in Fig. 1, e, f).

The lower attenuation with coaxial arrangement of resonators (Fig. 1, d, f) may be explained by radiation interference processes. The obtained dependencies demonstrate the possibility of transforming direct into reverse waves when detuning relative to the central frequency of the resonators (Fig. 1, e, g). It is also seen that increasing the distance between the centers of the resonators significantly changes the attenuation and frequency bands of the transmission line (Fig. 1, h).

Fig. 2 (c - f) shows the dependencies of the real and imaginary parts ψ on the frequency for one-dimensional lattices of DRs, in the region of their Whispering Gallery Mode oscillation $HE_{1,20,1}$, found using the approximate formula (10) (green lines) and by the exact solution of equation (9) (points in Fig. 2, c, d).

Obtained dependencies (9), (10) allow us using (7) to calculate the value of the group delay during a pulse transmission through one period of the line. Using the definition of a group velocity: $v_g = -1/[\partial \mathrm{Re}(\gamma)/\partial\omega]$. we define the delay time as:. $\Delta t_{1DR} = \Delta z / v_g$.

Neglecting the imaginary part γ, and using Cauchy-Riemann differentiability conditions, we find $v_g = -\mathrm{Re}(\partial\omega/\partial\gamma)$. From (9):

$$\Delta t_{1DR} = \mathrm{Re}\{[\omega_0 \sum_{s=1}^{\infty} \kappa_{|s|}\, s \cdot \sin(s\psi)]^{-1}\}. \qquad (11)$$

In the approximation of coupling only between adjacent resonators:

$$\Delta t_{1DR} \approx \mathrm{Re}[\frac{1}{\omega_0\ \kappa_1 \sin(\psi)}]. \qquad (12)$$

The obtained delay time relation (11) demonstrates the high sensitivity of the delay value to the distance between the resonators, due to the interaction of the DR fields. Fig. 2, b, h shows an example of calculating the time delay using formula (11) for oscillations $HE_{1,20,1}$ of cylindrical DRs.

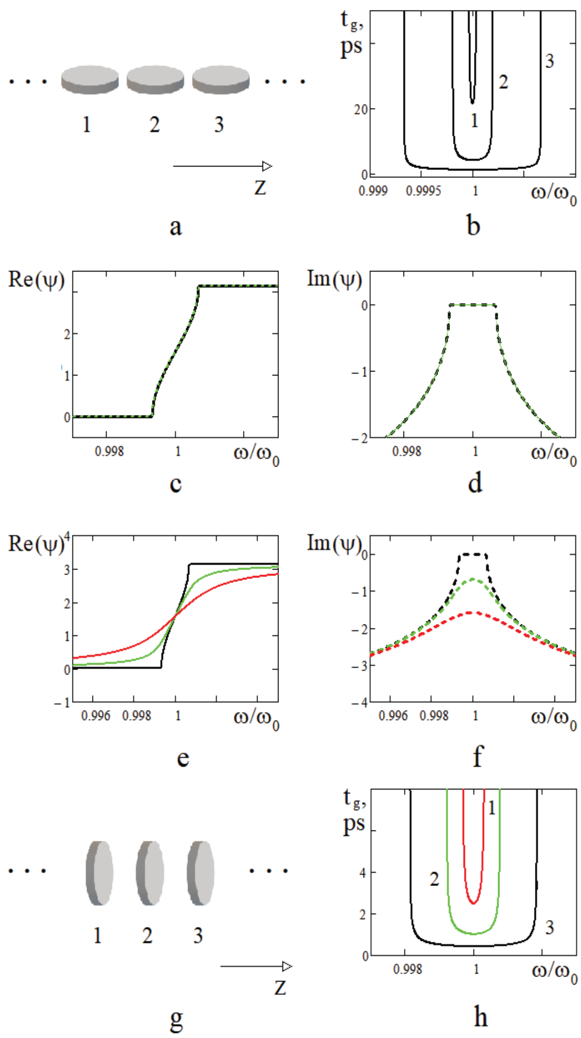

Fig. 2. Dependencies of the group delay (b); and of the real (c, e) and the imaginary (d, f) parts of the longitudinal wave parameter for $s = 0$ on the relative frequency for microresonators (a) with $HE_{1,20,1}$ oscillation; made of dielectric $\varepsilon_{1r} = 9{,}6$; relative sizes: $\Delta = L/2r_0 = 0{,}2$; the distance between centers of adjacent resonators: $k_0\Delta z = 17{,}15$ (9) (c – d); for $\tilde{k}_s^{\varepsilon} \approx \varepsilon''/\varepsilon' = 10^{-5}$ black lines; 10^{-3} green lines; $10^{-2,5}$ red lines (e - f). (b): $f_0 = 200$ THz; 1 - $k_0\Delta z = 2{,}15 \cdot q_\perp$; 2 - $k_0\Delta z = 2{,}08 \cdot q_\perp$; 3 - $k_0\Delta z = 2{,}03 \cdot q_\perp$ ($q_\perp = k_0 r_0$; r_0 - DR radius). (b, h): $f_0 = 200$ THz; (b): 1 - $k_0\Delta z = 2{,}15 \cdot q_\perp$; 2 - $k_0\Delta z = 2{,}08 \cdot q_\perp$; 3 - $k_0\Delta z = 2{,}03 \cdot q_\perp$ ($q_\perp = k_0 r_0$); (h): 1 - $k_0\Delta z = 2{,}9 \cdot q_z$; 2 - $k_0\Delta z = 3{,}1 \cdot q_z$; 3 - $k_0\Delta z = 3{,}3 \cdot q_z$ ($q_z = k_0 L/2$).

The dependence on the losses in the resonator dielectric ($\tilde{k}_s^\varepsilon \approx \varepsilon_s'' / \varepsilon_s'$) can be calculated by replacing $\tilde{k}_0$ in (8), (9) with $\tilde{k}_0 + \tilde{k}_s^\varepsilon$ [43] (Fig. 2, e – f).

IV. Coupled Oscillation Waves in Two-Dimensional Lattices of DRs

To calculate the parameters of waves propagating in two-dimensional resonator lattices, equations (2) were used, obtained under the assumption that coupling only between adjacent resonators of a rectangular structure was taken into account.

To obtain the solution in analytical form, we defined the "coordinates" of each resonator of the lattice by two integers (s,t), where s denotes the number of the position along the x axis, and the number t of the position of the DR along the axis z in a rectangular lattice (Fig. 3, a). The equation for the s,t-th DR, taking into account the coupling only with the nearest resonators, was written as:

$$\kappa_x(b_{s-1,t} + b_{s+1,t}) + \kappa_z(b_{s,t-1} + b_{s,t+1}) + (i\tilde{k}_0 - \lambda)b_{s,t} +$$

$$+\kappa_{xz}(b_{s-1,t-1} + b_{s-1,t+1} + b_{s+1,t-1} + b_{s+1,t+1}) = 0 \qquad (13)$$

Here we have designated: $\kappa_{s-1,t|s,t} = \kappa_{s+1,t|s,t} = \kappa_x$; $\kappa_{s-1,t-1|s,t} = \kappa_{s-1,t+1|s,t} = \kappa_{s+1,t-1|s,t} = \kappa_{s+1,t+1|s,t} = \kappa_{xz}$; $\kappa_{s,t-1|s,t} = \kappa_{s,t+1|s,t} = \kappa_z$. Where $\kappa_{u,v|s,t}$ - the mutual coupling coefficients between the u,v and s,t resonator.

The solution of system (13) was also expressed in the form of waves propagating in the direction of the axis z:

$$b_{s,t} = b_0 \sin(\theta_x s) e^{\mp i\psi t}, \qquad (14)$$

Substituting (14) into (13), after simple transformations, we obtain an equation similar to (8):

$$2\kappa_x \cos(\theta_x) + 2\kappa_z \cos(\psi) + (i\tilde{k}_0 - \lambda) +$$

$$+4\kappa_{xz} \cos(\theta_x)\cos(\psi) = 0. \qquad (15)$$

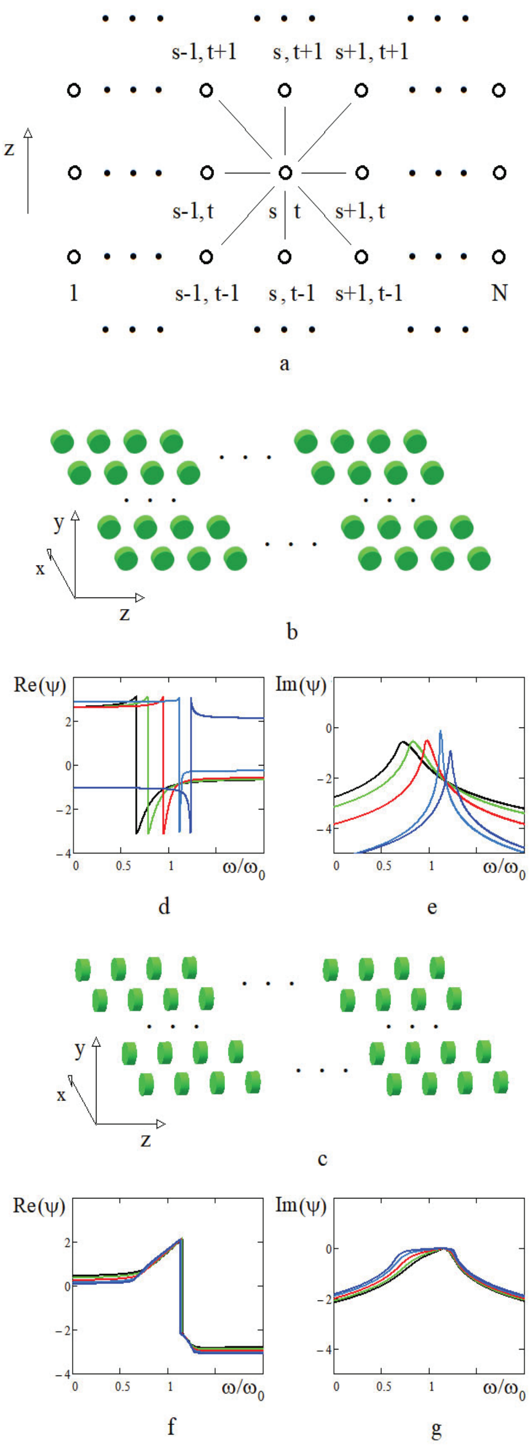

Fig. 3. 2-D infinite lattices (a, b, c) of DRs. Dependencies of the real (d, f) and imaginary (e, g) parts of the longitudinal wave parameter on the relative frequency for cylindrical resonators made of dielectric $\varepsilon_{1r} = 20$; relative sizes: $\Delta = L/2r_0 = 0,4$; $N = 5$; $s = 0$ (15); distance between the adjacent resonators: $k_0\Delta x = 1$; $k_0\Delta z = 2$ (d, e); $k_0\Delta x = 2$; $k_0\Delta z = 1$ (f, g)

Equation (15) was supplemented with symmetry conditions: $|b_{u.t}| = |b_{N-u+1.t}|$, from which we determined the N values:

$$\theta_x = \theta_x^n = \frac{n\pi}{(N+1)}; \quad (n = 1,2,...,N) \qquad (16)$$

Substituting (16) into (15), we get:

$$\lambda = i\tilde{k}_0 + 2\kappa_x \cos[\frac{n\pi}{(N+1)}] + . \qquad (17)$$

$$+2\left[\kappa_z + 2\kappa_{xz}\cos[\frac{n\pi}{(N+1)}]\right]\cos(\psi) .$$

It follows that in the direction of the axis z of a two-dimensional lattice consisting of N resonators in cross section, only N types different "eigen" modes can propagate. Each such mode is characterized by its own distribution of resonator amplitudes in the transverse plane $z = \text{const}$, determined by a given number, $n = 1,2,...,N$. For each value of the n, the characteristic equation (17) determines the degenerate on s dependence of the longitudinal number ψ on the frequency in the region of natural oscillations of the partial DR:

$$\psi \approx \psi^{ns} = \pm\arccos\{\left[\frac{\omega}{\omega_0} - 1 - i\frac{\tilde{k}_0}{2} - \kappa_x \cos[\frac{n\pi}{(N+1)}]\right] / .$$

$$/\left[\kappa_z + 2\kappa_{xz}\cos[\frac{n\pi}{(N+1)}]\right]\} \pm 2s\pi; \quad (18)$$

$$(n = 1,2,...,N; s = 0,1,2,...,\infty)$$

Equation (18) also needs to be supplemented by the condition: $\operatorname{Im}(\psi) \le 0$,

Fig. 3 shows the dependencies of complex values ψ on the frequency in the region of fundamental oscillations of resonators H_{101} of an infinite rectangular lattice of $N = 5$ cylindrical DRs, calculated using (18) for $s = 0$.

The obtained solutions demonstrate that in the case of the DR arrangement shown in Fig. 3, b. the mode composition of the waves can change significantly, both due to a change ψ in the real part and due to different attenuation values (Fig. 3, b. d, e).

In the general case, if the lattice dimensions are not limited in two directions, the solution of the system (13) obviously takes the form:

$$b_{s,t} = b_0 e^{\mp i\psi_x s \mp i\psi_z t}, \qquad (19)$$

in which the complex constants ψ_x, ψ_z, in the directions x, z respectively, are related to each other by a dispersion equation:

$$1 + i\tilde{k}_0/2 + \kappa_x\cos(\psi_x) + \kappa_z\cos(\psi_z) +$$

$$+2\kappa_{xz}\cos(\psi_x)\cos(\psi_z) = \omega/\omega_0 . \qquad (20)$$

V. Waves in Three-Dimensional Lattices of DRs

In a similar manner, the parameters of the waves of a three-dimensional rectangular lattice, consisting of $N \times M$ identical resonators in cross section, were calculated (Fig. 4). To construct an analytical solution to the system of equations (2), we also defined the coordinates of each resonator by three integer indices (s,t,u), where s denotes the position along the axis x, t denotes the position along the axis y, and u denotes the DR position along the axis z. If all resonators are identical, then the mutual coupling coefficients are symmetrical and for a rectangular lattice:

$$\kappa_{s-1,t,u|s,t,u} = \kappa_{s+1,t,u|s,t,u} = \kappa_x ;$$

$$\kappa_{s,t-1,u|s,t,u} = \kappa_{s,t+1,u|s,t,u} = \kappa_y ;$$

$$\kappa_{s,t,u-1|s,t,u} = \kappa_{s,t,u+1|s,t,u} = \kappa_z ;$$

$$\kappa_{s-1,t-1,u|s,t,u} = \kappa_{s-1,t+1,u|s,t,u} =$$

$$= \kappa_{s+1,t-1,u|s,t,u} = \kappa_{s+1,t+1,u|s,t,u} = \kappa_{xy} ;$$

$$\kappa_{s-1,t,u-1|s,t,u} = \kappa_{s-1,t,u+1|s,t,u} =$$

$$= \kappa_{s+1,t,u-1|s,t,u} = \kappa_{s+1,t,u+1|s,t,u} = \kappa_{xz} ;$$

$$\kappa_{s,t-1,u-1|s,t,u} = \kappa_{s,t-1,u+1|s,t,u} =$$

$$= \kappa_{s,t+1,u-1|s,t,u} = \kappa_{s,t+1,u+1|s,t,u} = \kappa_{yz} ;$$

$$\kappa_{s-1,t-1,u-1|s,t,u} = \kappa_{s+1,t+1,u+1|s,t,u} =$$

$$= \kappa_{s+1,t-1,u-1|s,t,u} = \kappa_{s-1,t+1,u+1|s,t,u} = \kappa_{xyz} ;$$

$$\kappa_{s-1,t+1,u-1|s,t,u} = \kappa_{s+1,t-1,u+1|s,t,u} =$$

$$= \kappa_{s+1,t+1,u-1|s,t,u} = \kappa_{s-1,t-1,u+1|s,t,u} = \kappa_{xyz} .$$

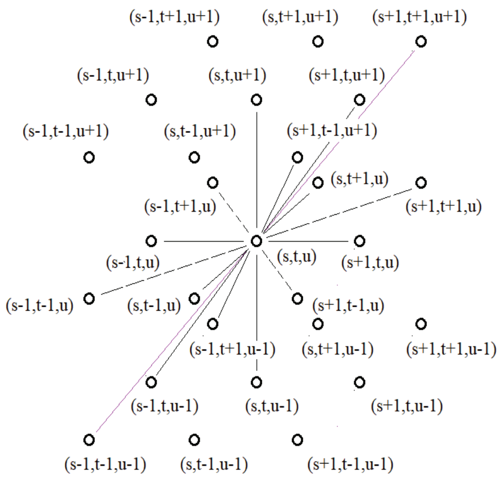

Fig. 4. Three-dimensional rectangular lattice of DR.

Then equation (5) for the selected DR with "coordinates" (s,t,u), taking into account the coupling only with the nearest resonators of the lattice (Fig. 4), was rewritten in the form:

$$\kappa_x(b_{s-1,t,u} + b_{s+1,t,u}) + \kappa_y(b_{s,t-1,u} + b_{s,t+1,u}) +$$

$$+\kappa_z(b_{s,t,u-1} + b_{s,t,u+1}) + (i\tilde{k}_0 - \lambda)b_{s.t,u} +$$

$$+\kappa_{xy}(b_{s-1,t-1,u} + b_{s+1,t+1,u} + b_{s+1,t-1,u} + b_{s-1,t+1,u}) +$$

$$+\kappa_{xz}(b_{s-1,t,u-1} + b_{s+1,t,u-1} + b_{s-1,t,u+1} + b_{s+1,t,u+1}) +$$

$$+\kappa_{yz}(b_{s,t-1,u-1} + b_{s,t+1,u-1} + b_{s,t-1,u+1} + b_{s,t+1,u+1}) + \quad (21)$$

$$+\kappa_{xyz}(b_{s-1,t-1,u-1} + b_{s+1,t+1,u+1} + b_{s+1,t-1,u-1} + b_{s-1,t+1,u+1} +$$

$$+b_{s-1,t+1,u-1} + b_{s+1,t-1,u+1} + b_{s+1,t+1,u-1} + b_{s-1,t-1,u+1}) = 0 .$$

For an infinite lattice along the axis z, the solution of the equations system (21) was sought in the form of "plane" waves of the DR amplitudes:

$$b_{s.t} = b_0 \sin(\theta_x s)\sin(\theta_y t)e^{\mp i\psi u} , \quad (22)$$

Substituting (22) into (21), we obtain:

$$\lambda = i\tilde{k}_0 + 2\kappa_x\cos(\theta_x) + 2\kappa_y\cos(\theta_y) + 2\kappa_z\cos(\psi) +$$

$$+4\kappa_{xy}\cos(\theta_x)\cos(\theta_y) + 4\kappa_{xz}\cos(\theta_x)\cos(\psi) + \quad (23)$$

$$+4\kappa_{yz}\cos(\theta_y)\cos(\psi) + 8\kappa_{xyz}\cos(\theta_x)\cos(\theta_y)\cos(\psi)$$

Based on the requirement to fulfil the conditions of field symmetry in the "transverse plane" of the waveguide structure:

$$\left|b_{w.t}\right| = \left|b_{N-w+1.t}\right|; \ \left|b_{s.v}\right| = \left|b_{s.M-v+1}\right|;$$

we found:

$$\theta_x = \theta_x^n = \frac{n\pi}{(N+1)}; \ \ \theta_y = \theta_y^m = \frac{m\pi}{(M+1)}; \qquad (24)$$

where $n = 1,2,...,N$; $m = 1,2,...,M$.

From here we again came to the conclusion that in an infinite lattice, the cross-section of which represents $N \times M$ a rectangular DR structure, only $N \times M$ independent degenerate "plane" waves can propagate in the direction of the axis z, each of which is characterised by a given distribution of the amplitudes of the coupled oscillations of the resonators in the transverse plane of the line.

For each such wave, the approximate dependence of the longitudinal wave parameter on the frequency is valid:

$$\psi \approx \pm \arccos\Bigg\{\left[\frac{\omega}{\omega_0} - 1 - i\frac{\tilde{k}_0}{2} - \kappa_x \cos[\frac{n\pi}{(N+1)}] - \right.$$

$$\left. -\kappa_y \cos[\frac{m\pi}{(M+1)}] - \kappa_{xy} \cos[\frac{n\pi}{(N+1)}]\cos[\frac{m\pi}{(M+1)}]\right] /$$

$$/\left[\kappa_z + 2\kappa_{xz}\cos[\frac{n\pi}{(N+1)}] + 2\kappa_{yz}\cos[\frac{m\pi}{(M+1)}] + . \right.$$

$$\left. + 4\kappa_{xyz}\cos[\frac{n\pi}{(N+1)}]\cos[\frac{m\pi}{(M+1)}]\right]\Bigg\} \pm 2s\pi \qquad (25)$$

$$(s = 0,1,2,...,\infty)$$

Equation (25) also must be supplemented with the condition: $\mathrm{Im}(\psi) \le 0$.

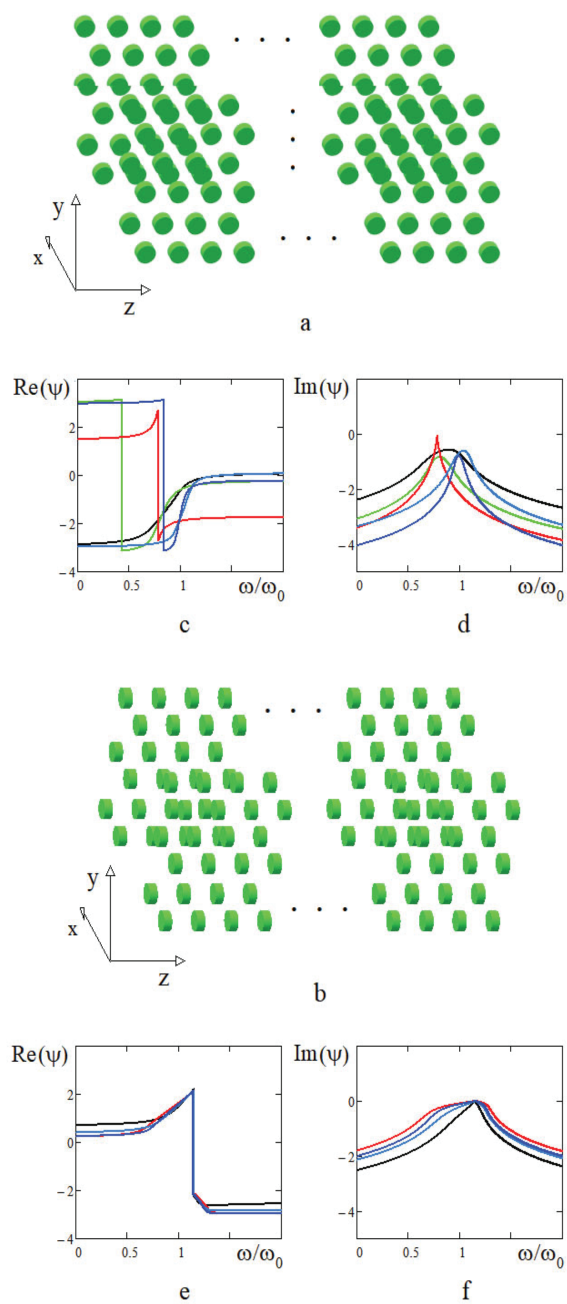

Fig. 5. Three-dimensional infinite rectangular lattices (a, b) of cylindrical DRs. Dependencies of the real (c, e) and imaginary (d, f) parts of the longitudinal wave parameter on the relative frequency for cylindrical resonators made of dielectric $\varepsilon_{1r} = 20$; relative sizes: $\Delta = L/2r_0 = 0,4$; $N \times M = 3 \times 3$ DR; $s = 0$ (22); distance between the adjacent resonators: $k_0\Delta x = 1$; $k_0\Delta y = 2$; $k_0\Delta z = 2,5$ (c, d); . $k_0\Delta x = 2$; $k_0\Delta y = 2$; $k_0\Delta z = 1$ (e, f).

The generalisation of the developed theory to cases of wave propagation in arbitrary directions of an unlimited three-dimensional lattice is obvious: the solution of system (21) should be sought in the form :

$$\mathrm{b}_{s.t,u} = \mathrm{b}_0 e^{\mp i\psi_x s \mp i\psi_y t \mp i\psi_z u}, \qquad (26)$$

and the corresponding dispersion equation, obtained from the system of equations (21), must connect the wave parameters:

$$\omega / \omega_0 = i\tilde{k}_0 / 2 + \kappa_x \cos(\psi_x) + \kappa_y \cos(\psi_y) + \kappa_z \cos(\psi_z) +$$

$$+2\kappa_{xy} \cos(\psi_x)\cos(\psi_y) + 2\kappa_{xz} \cos(\psi_x)\cos(\psi_z) +$$

$$(27)$$

$$+2\kappa_{yz} \cos(\psi_y)\cos(\psi_z) + 4\kappa_{xyz} \cos(\psi_x)\cos(\psi_y)\cos(\psi_z)$$

.

Fig. 5 shows the dependencies of the wave parameters on the frequency for $\mathrm{N} \times \mathrm{M} = 3 \times 3$ a rectangular DR lattice of cylindrical shape, excited on the main types of natural oscillations H_{101} for $\mathrm{s} = 0$ (25).

It is obvious that an increase in the dimensionality of the spatial structure leads to an increase in the number of natural waves and frequency distribution, caused by an increase in the number of couplings between resonators.

VI. Three-dimensional waveguide lattices of DRs

Three-dimensional structures of coupled resonators allow forming spatial waveguide structures of various cross-section shapes from them. We have considered several examples of calculating the characteristics of an axially symmetric waveguide structure consisting of an infinite number of ring lattices with the same number of resonators (Fig. 6, a, b).

Initially, we considered an axially symmetric ring lattice with the same number of DRs (Fig. 6, a), which allows an analytical description of coupled oscillations in a general form. Let us have a linear structure of ring lattices with the same number of resonators N. All sublattices are located axially symmetrically relative to the selected common axis z.

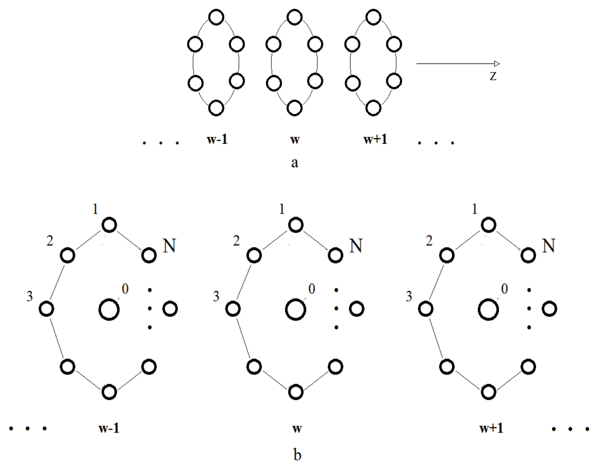

Fig. 6. Three-dimensional cylindrical lattices of identical (a); different (b) resonators

All eigenvectors of the ring lattices are the same for the same number of resonators [44], however, for convenience, we have designated each of the vectors of the w-th lattice by the index w:

$$\mathbf{b}_o^j = \mathbf{b}_o^{wj} = \begin{pmatrix} \mathrm{b}_1^{wj} \\ \mathrm{b}_2^{wj} \\ \vdots \\ \mathrm{b}_N^{wj} \end{pmatrix} = \frac{1}{\sqrt{\mathrm{N}}} \cdot \begin{pmatrix} 1 \\ \eta_j^1 \\ \vdots \\ \eta_j^{N-1} \end{pmatrix},$$

$$(w = -\infty, ..., +\infty;\ j = 0, 2, ..., N-1) \qquad (28)$$

here $\eta_j = \exp(2 j\pi i / \mathrm{N})$ j-th of Nth complex root of unity.

j-th eigenvector (28) of the isolated w-th ring sublattice with N resonators satisfies the equation:

$$(i\tilde{k}_w - \lambda_o^{wj})b_o^{sj} + \sum_{n=1}^{N-1} \kappa_{ns}^w \, b_o^{nj} = 0\,; \qquad (29)$$

$$(w = 1,2,...,M)$$

where

$$\lambda_o^{wj} = i\tilde{k}_w + \sum_{v=1}^{N-1} \kappa_v^w \cdot (\eta_j)^v\,,\ \ (j = 0,1,...,N-1) \qquad (30)$$

The axially symmetric arrangement of sublattices with the same number of resonators also allowed us to reduce the total number of mutual coupling coefficients.

Taking into account the rotational symmetry of the resonators ($\kappa_{tu}^{sw} = \kappa_{t-u}^{sw}$) of the system under consideration, we determined the "vector" of mutual coupling between the u-th and w-th sublattices:

$$\mathbf{K}_{uw} = \begin{pmatrix} \kappa_0^{uw} \\ \kappa_1^{uw} \\ \vdots \\ \kappa_{N-1}^{uw} \end{pmatrix}; \qquad (31)$$

as well as the vector of mutual coupling of the resonators in the w-th sublattices:

$$\mathbf{K}_{w} = \begin{pmatrix} \kappa_1^{w} \\ \kappa_2^{w} \\ \vdots \\ \kappa_{N-1}^{w} \end{pmatrix}. \qquad (32)$$

Next we regrouped the terms in equations (2) into two partial sums, the first of which related to the oscillations of the selected w-th sublattice and the sum related to the remaining sublattices:

$$(i\tilde{k}_w - \lambda)b_t + \sum_{n=1}^{N-1} \kappa_n^w \, b_n + \sum_{u \neq w = -\infty}^{+\infty} \sum_{v=0}^{N-1} \kappa_{vt}^{uw} \, b_v = 0 \qquad (33)$$

The solution to the system of equations (33) for the j-th type of oscillations of the resonators of the w-th sublattice was sought in the form:

$$\mathbf{b}^{(j)} = a^w \cdot \mathbf{b}_o^j\,; \qquad (34)$$

Multiplying equation (29) by a^w and subtracting it from (33), we get:

$$(\lambda_o^{wj} - \lambda)a^w \cdot b_o^{tj} + \sum_{u \neq w = -\infty}^{+\infty} [\sum_{v=0}^{N-1} \kappa_{vt}^{uw} \, b_o^{vj}] \cdot a^u = 0\,. \ (35)$$

Since the sum $\sum_{v=0}^{N-1} \kappa_{vt}^{uw} b_v$ is taken over all resonators of the u-th sublattice, it does not depend on the initial value of the index and can be rewritten as:

$$\sum_{v=0}^{N-1} \kappa_{vt}^{uw} \, b_o^{vj} = \sum_{v=0}^{N-1} \kappa_{(t+v)t}^{uw} \, b_o^{(t+v)j}\,, \qquad (36)$$

Substituting (36) into (35), we found:

$$(\lambda_o^{wj} - \lambda)a^w \cdot b_o^{tj} + \sum_{u \neq w = -\infty}^{+\infty} [\sum_{v=0}^{N-1} \kappa_{(t+v)t}^{uw} \, b_o^{(t+v)j}] \cdot a^u = 0\,. \qquad (37)$$

Dividing (37) by b_o^{tj}, and taking into account (28):

$$b_o^{(v+t-1)j} / b_o^{tj} = \sqrt{N} b_o^{(v)j}\,,$$

we ended up with an equation that does not depend on the t-th DR,

$$(\lambda_o^{wj} - \lambda)a^w + \sqrt{N} \sum_{u \neq w = -\infty}^{+\infty} [\sum_{v=1}^{N} \kappa_{v-1}^{uw} \, b_o^{(v)j}] \cdot a^u = 0\,, \quad (38)$$

determined only by the parameters of the ring sublattices. Here we also used the condition $\kappa_{qv}^{uw} = \kappa_{q-v}^{uw}$

Equation (38) can be rewritten in a more compact form, taking into account definition (31):

$$(\lambda_o^{wj} - \lambda) \cdot a^w + \sqrt{N} \sum_{u \neq w = -\infty}^{+\infty} (\mathbf{K}_{uw}, \mathbf{b}_o^j) \cdot a^u = 0\,. \qquad (39)$$

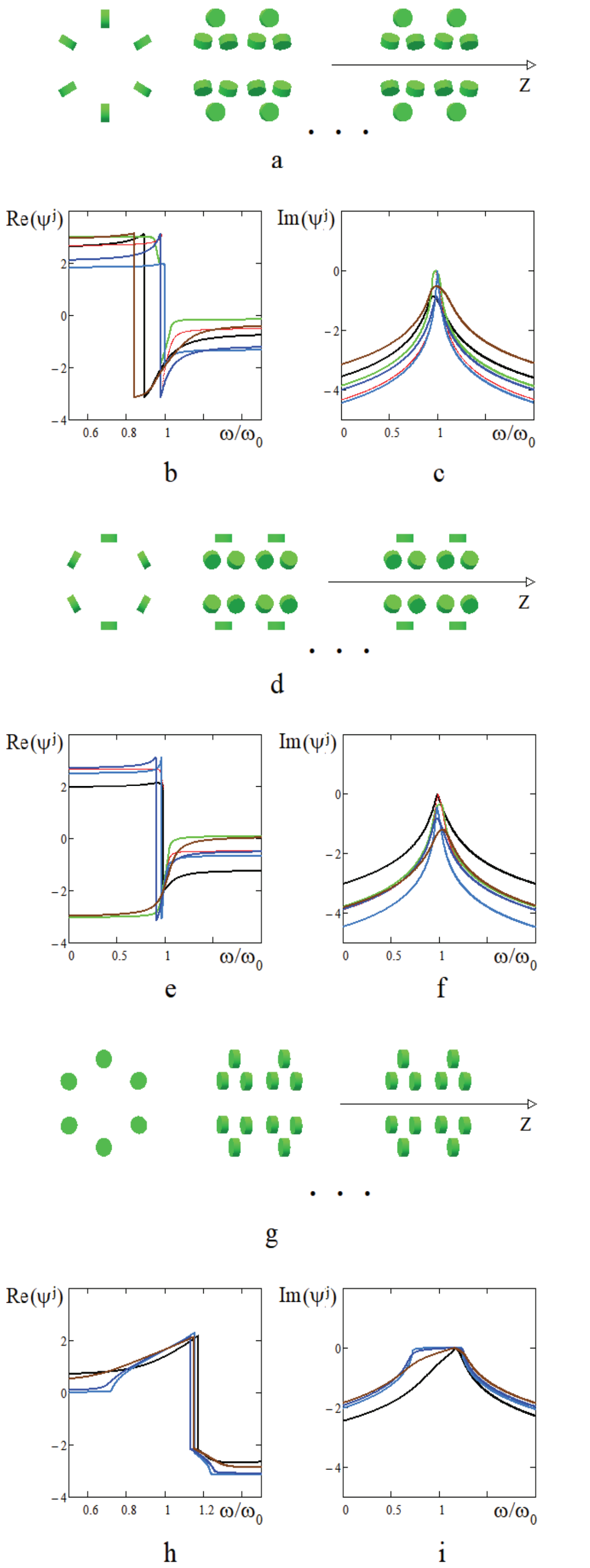

Fig. 7. Three-dimensional cylindrical lattice (a) of identical resonators in the position A (a); B (d); C (g). $\varepsilon_{1r} = 20$; relative sizes: $\Delta = L/2r_0 = 0{,}4$; $N = 6$; $s = 0$; $k_0\Delta r = 2$; $k_0\Delta z = 2$ (b, c; e, f); $k_0\Delta r = 1{,}6$; $k_0\Delta z = 1$ (h, i).

The infinite system of equations (39) in general form determines the frequencies and amplitudes of coupled oscillations of periodic waveguide structures of axially symmetric ring sublattices with the same number N of resonators.

From (39) it also follows that the coupling vectors of the ring sublattices (32) are not included in the system in an explicit form. They determine the frequencies of natural oscillations of the sublattices. λ_o^{wj}.

By analogy with the oscillations of individual resonators (4), we may introduce into consideration the infinite coupling matrices of the ring sublattices:

$$K^{(j)} = \qquad (40)$$

$$= \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \lambda_o^{w-1j} & \sqrt{N}(\mathbf{K}_{ww-1}, \mathbf{b}_o^j) & \sqrt{N}(\mathbf{K}_{w+1w-1}, \mathbf{b}_o^j) & \dots \\ \dots & \sqrt{N}(\mathbf{K}_{w-1w}, \mathbf{b}_o^j) & \lambda_o^{wj} & \sqrt{N}(\mathbf{K}_{w+1w}, \mathbf{b}_o^j) & \dots \\ \dots & \sqrt{N}(\mathbf{K}_{w-1w+1}, \mathbf{b}_o^j) & \sqrt{N}(\mathbf{K}_{ww+1}, \mathbf{b}_o^j) & \lambda_o^{w+1j} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

the eigenvalues and eigenvectors of which, together with (30), determine the parameters of the structure.

From this it follows that if all sublattices are no interact, $\mathbf{K}_{uw} = 0$, the eigenvalues of the matrix $K^{(j)}$ are determined by the set of eigenvalues of isolated ring sublattices λ_o^{wj} (30).

In the case of identical resonators in all sublattices, the eigenvalues of matrix (40) are determined by the expression:

$$\lambda^j = \lambda_o^j + \bar{\lambda}^j,$$

where $\lambda_o^{wj} = \lambda_o^j$, and $\bar{\lambda}^j$ - eigenvalue of a matrix:

$$\bar{K}^{(j)} = \qquad (41)$$

$$= \begin{pmatrix} . & ... & ... & \cdots & . \\ . & 0 & \sqrt{N}(\mathbf{K}_{ww-1},\mathbf{b}_o^j) & \sqrt{N}(\mathbf{K}_{w+1w-1},\mathbf{b}_o^j) & . \\ . & \sqrt{N}(\mathbf{K}_{w-1w},\mathbf{b}_o^j) & 0 & \sqrt{N}(\mathbf{K}_{w+1w},\mathbf{b}_o^j) & . \\ . & \sqrt{N}(\mathbf{K}_{w-1w+1},\mathbf{b}_o^j) & \sqrt{N}(\mathbf{K}_{ww+1},\mathbf{b}_o^j) & 0 & . \\ . & ... & ... & \cdots & . \end{pmatrix}$$

For a infinite number of sublattices, with the same DR, we seek the solution of system (39) in the form:

$$a^w = a_0 e^{\mp i\psi w}, \qquad (42)$$

where a_0 and ψ are constant also do not depend on the sublattice numbers.

Substituting (42) into (38), after simple transformations and reduction of amplitudes a_0, we find:

$$\lambda^j = \lambda_o^j + 2\sqrt{N}\sum_{u=1}^{+\infty}(\mathbf{K}_{1u+1},\mathbf{b}_o^j)\cos(u\psi). \qquad (43)$$

Equation (43) together with (30) and the condition $\text{Im}(\psi) \le 0$, also determine the N sets of dispersion functions of the waves that may be propagated in the ring structures of resonators.

In the approximation of taking into account the coupling only between adjacent sublattices:

$$\lambda^j \approx \lambda_o^j + 2\cos(\psi)\sqrt{N}(\mathbf{K}_{12},\mathbf{b}_o^j)$$

or

$$\psi(\omega) \approx \pm\arccos\left[\frac{2(\omega/\omega_0 - 1) - \lambda_o^j}{2\sqrt{N}(\mathbf{K}_{12},\mathbf{b}_o^j)}\right] \pm 2s\pi. \qquad (44)$$

Where λ_o^j defined by (30).

Fig. 7 shows the frequency dependencies of the wave parameters of ring, axially symmetric lattices with cylindrical DRs of different spatial orientations. The obtained data show a significantly smaller spread of modal parameters. Unlike the “solid” lattices considered in III–V, hollow waveguide structures are characterized by a lower in-band and a higher attenuation value outside the transmission frequencies. (Fig. 7, c, f, i).

We considered also an infinite linear structure of identical coaxial sublattices, shown in Fig. 6, b, each of which consists of a $N+1$ DR. We designated the resonators of each sublattice (Fig. 6, b) by numbers $s = 0,1,2,...,N$; resonators located on the axis z of the sublattices was designated by indices 0, while resonators located on a ring of the sublattice, was designated by $s = 1,2,...,N$. The distance between resonators of the adjacent sublattices we also designed by Δz.

We denoted the amplitudes of the resonators by b_t^w, where t - is the resonator number, w - number of the sublattice. The mutual coupling coefficients between s-th and t-th resonators in the w-th sublattice we denoted by κ_{st}^w; the mutual coupling coefficients between s-th resonator of u-th sublattice and t-th resonators in the w-th sublattice we denoted by κ_{st}^{uw} ($u \ne w$). The coupling coefficients of the axial resonators with the external structure are denoted by $\tilde{k}_0$, and the coupling coefficients of the resonators with the external structure located on the ring sublattices was denoted by $\tilde{k}_1$.

We broke down the sum included in the system of equations (2) into terms related to the w-th sublattice with the selected t-th resonator and to the remaining sublattices:

$$(i\tilde{k}_t - \lambda)b_t^w + \sum_{s=1,s\ne t}^{N}\kappa_{st}^w\, b_s^w + \sum_{u=-\infty,u\ne w}^{+\infty}\sum_{s=0}^{N}\kappa_{st}^{uw}\, b_s^u = 0. \quad (45)$$

We explicitly selected in (45) the amplitudes of the DR related to the resonators located on the axis z:

$$(i\tilde{k}_0 - \lambda)b_0^w + \sum_{s=1}^{N}\kappa_{s0}^w\, b_s^w + \sum_{u=-\infty,u\ne w}^{+\infty}\kappa_{00}^{uw}b_0^u + $$
$$+ \sum_{u=-\infty,u\ne w}^{+\infty}\sum_{s=1}^{N}\kappa_{s0}^{uw}\, b_s^u = 0;$$

(46)

$$(i\tilde{k}_1 - \lambda)b_t^w + \kappa_{0t}^w b_0^w + \sum_{s=1,s\ne t}^{N}\kappa_{st}^w\, b_s^w + \sum_{u=-\infty,u\ne w}^{+\infty}\kappa_{0t}^{uw}b_0^u +$$

$$+\sum_{u=-\infty,u\neq w}^{+\infty}\sum_{s=1}^{N}\kappa_{st}^{uw}\,b_s^u=0\,.$$

For resonators located on the axis z, we considered only oscillations whose fields are azimuthally symmetrical. For such oscillations, the coefficients of mutual coupling κ_{s0}^{w} and κ_{s0}^{uw} are the same and do not depend on s: $\kappa_{s0}^{w}=\kappa_{10}^{w}$; $\kappa_{s0}^{uw}=\kappa_{10}^{uw}$.

In this case, two subsets of solutions can be defined in (46):

1) **Ring sublattices oscillations** with $b_0^w=0$:

$$\kappa_{10}^{w}\sum_{s=1}^{N}b_s^w+\sum_{u=-\infty,u\neq w}^{+\infty}\kappa_{10}^{uw}\sum_{s=1}^{N}b_s^u=0\,.$$

$$(i\tilde{k}_1-\lambda)b_t^w+\sum_{s=1,s\neq t}^{N}\kappa_{st}^{w}\,b_s^w+\sum_{u=-\infty,u\neq w}^{+\infty}\sum_{s=1}^{N}\kappa_{st}^{uw}\,b_s^u=0 \quad (47)$$

2) **Azimuthally uniform oscillations**: $b_s^w=b_1^w$:

$$(i\tilde{k}_0-\lambda)b_0^w+\sum_{u=-\infty,u\neq w}^{+\infty}\kappa_{00}^{uw}b_0^u+b_1^w\sum_{s=1}^{N}\kappa_{s0}^{w}+$$
$$+\sum_{u=-\infty,u\neq w}^{+\infty}b_1^u\sum_{s=1}^{N}\kappa_{s0}^{uw}=0\,;$$
$$(48)$$
$$(i\tilde{k}_1-\lambda)b_1^w+\kappa_{01}^{w}b_0^w+b_1^w\sum_{s=1,s\neq 1}^{N}\kappa_{s1}^{w}+\sum_{u=-\infty,u\neq w}^{+\infty}\kappa_{01}^{uw}b_0^u+$$
$$+\sum_{u=-\infty,u\neq w}^{+\infty}b_1^u\sum_{s=1}^{N}\kappa_{s1}^{uw}=0\,.$$

We will seek the solution to the system of equations (47) in the form (34):

$$b_s^w=a^w b_o^{sj}\,; \qquad (49)$$

under an additional condition: $1\le j\le N-1$. In this case, as it`s follows from (28): $\sum_{s=1}^{N}b_s^u=a^u\sum_{s=1}^{N}b_o^{sj}=0$. Than the first equations of the system (47) are satisfied. The second equation of the system coincides with (33), the solutions of which we have already found (42) – (44). These solutions coincide with the solutions for the ring lattices for $1\le j\le N-1$.

For azimuthally uniform oscillations, further simplification of equations (48) is possible if all resonators of each sublattice are located periodically along the axis z. In this case, the solution can be represented as:

$$b_{0,1}^{w}=b_{0,1}e^{\mp i\psi w}\,, \qquad (50)$$

The complex parameter ψ, is also not function on w.

Substituting (50) into (48), we obtained, provided that the coupling between adjacent sublattices is taken into account and for only the azimuthal symmetry of the coupling coefficients:

$$\begin{cases}[(i\tilde{k}_0-\lambda)+2\cos(\psi)\kappa_{00}^{12}]b_0+N[\kappa_{10}^{1}+2\cos(\psi)\kappa_{10}^{12}]b_1=0\\ [\kappa_{01}^{1}+2\cos(\psi)\kappa_{01}^{12}]b_0+\\ +[(i\tilde{k}_1-\lambda)+\sum_{s=2}^{N}\kappa_{s1}^{1}+2\cos(\psi)\sum_{s=1}^{N}\kappa_{s1}^{12}]b_1=0\end{cases} \quad (51)$$

The solution of the system (51) is:

$$\psi^{\pm}=\arccos\left\{\frac{F^{\pm}}{4[\kappa_{00}^{12}\sum_{s=1}^{N}\kappa_{s1}^{12}-N\kappa_{10}^{12}\kappa_{01}^{12}]}\right\}\pm 2s\pi\,, \quad (52)$$

where

$$F^{\pm}=N(\kappa_{10}^{12}\kappa_{01}^{1}+\kappa_{01}^{12}\kappa_{10}^{1})-\kappa_{00}^{12}[(i\tilde{k}_1-\lambda)+$$
$$+\sum_{s=2}^{N}\kappa_{s1}^{1}]-(i\tilde{k}_0-\lambda)\sum_{s=1}^{N}\kappa_{s1}^{12}\pm d\,;$$

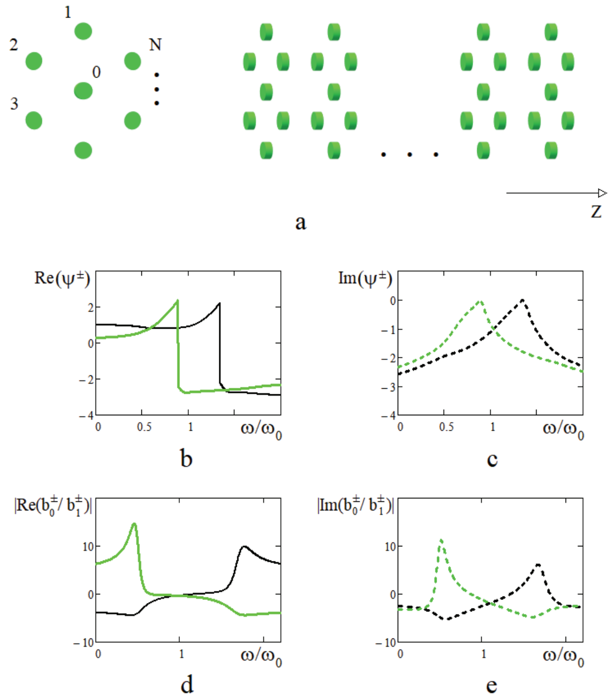

Fig. 8. Three-dimensional coaxial lattice (a) of identical DRs with H_{101} oscillations. (b – e): $\varepsilon_{1r} = 20$; $\Delta = L/2r_0 = 0,4$; $N = 24$; $s = 0$; $k_0\Delta z = 1,5$; distance between adjacent resonators of ring sublattice: $k_0\Delta r = 3$.

$$d = \left\{ \{N(\kappa^{12}_{10}\kappa^{1}_{01} + \kappa^{12}_{01}\kappa^{1}_{10}) - \kappa^{12}_{00}[(i\tilde{k}_1 - \lambda) + \sum_{s=2}^{N}\kappa^{1}_{s1}] - (i\tilde{k}_0 - \lambda)\sum_{s=1}^{N}\kappa^{12}_{s1}\}^2 - 4[\kappa^{12}_{00}\sum_{s=1}^{N}\kappa^{12}_{s1} - N\kappa^{12}_{10}\kappa^{12}_{01}] \cdot \cdot\{(i\tilde{k}_0 - \lambda)[(i\tilde{k}_1 - \lambda) + \sum_{s=2}^{N}\kappa^{1}_{s1}] - N\kappa^{1}_{10}\kappa^{1}_{01}\} \right\}^{1/2}.$$

And the amplitude ratio:

$$\frac{b_0}{b_1} = -N\frac{\kappa^{1}_{10} + 2\cos(\psi)\kappa^{12}_{10}}{(i\tilde{k}_0 - \lambda) + 2\cos(\psi)\kappa^{12}_{00}}. \tag{53}$$

Equation (52) also must be supplemented with the condition: $\mathrm{Im}(\psi) \le 0$.

In the particular case where $\kappa^{1}_{10} = 0$ and $\kappa^{12}_{10} = 0$, the axial and ring sublattices of the resonators oscillate independently of each other. In this case, for the axial sublattice the dispersion dependence takes the form (9), and for ring sublattices (44).

Fig. 8 shows the frequency dependencies of the wave parameters of coaxial lattices cylindrical DRs calculated from (52) – (53) for axially symmetrical oscillations. The obtained data demonstrate the existence of frequency regions in which the main power is transmitted through the central lattice of resonators (Fig.8, d, e).

VII. Conclusion

A perturbation theory to describe the processes of waves propagating in the lattices of DR that are not limited in one or several directions has been developed.

General systems of equations, describing wave processes in an infinite structure of axial ring lattices with the same number of resonators, are derived.

General analytical solutions for the frequency dependencies of the amplitudes of DR of one-, two- and three-dimensional waveguide structures are obtained.

The obtained general expressions allow us to calculate the delay time of pulse propagation in various transmission lines built on DR.

The developed theory serves as a basis for constructing a wide class of optical devices for different communication systems.

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Трубін О.О.

Теорія спрямованих хвиль в нескінченних системах зв’язаних діелектричних резонаторів

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Проблематика. Одним із багатообіцяючих елементів систем оптичного та квантового зв’язку є різноманітні лінії затримки, побудовані на високо добротних діелектричних резонаторах. Такі лінії зазвичай складаються з великого числа елементів тому оптимізація їх параметрів призводить до значних труднощів. Теорія діелектричних резонаторів є основою для розуміння, розрахунків та оптимізації параметрів ліній затримки та інших пристроїв, яка дозволяє суттєво скоротити розрахункові ресурси, які потребують використання потужних комп’ютерів.

Мета досліджень. Метою даного дослідження є отримання аналітичних виразів дисперсійних залежностей та розподілів електромагнітних полів різних видів ліній передачі, які складаються з великої кількості діелектричних резонаторів з ціллю використання їх в різноманітних пристроях оптичного зв’язку. Для вирішення цієї задачі виводиться нескінчена лінійна система рівнянь, отримана із теорії збурень для рівнянь Максвеллу, яка пов’язує між собою комплексні амплітуди, хвилеві числа та частоти резонаторів.

Методика реалізації. Для пошуку аналітичних виразів використовуються методи теорії збурень та теорія нескінчених лінійних рівнянь. Кінцевим результатом є нові загальні аналітичні формули для опису дисперсійних кривих решіток, які складаються із нескінченого числа діелектричних резонаторів різних видів.

Результати досліджень. Розвинута теорія розповсюдження хвиль в системах зв’язаних між собою одно-, двох- та трьох-вимірних решіток діелектричних резонаторів нескінчених в одному або декількох напрямках. Отримані нові аналітичні вирази для дисперсійних залежностей власних хвиль, часу затримки, а також розподілів комплексних амплітуд резонаторів, без обмеження на їх кількість. За допомогою теорії збурень, побудована нова аналітична модель, яка описує власні хвилі три-вимірних решіток, які складаються із однакових кільцевих структур діелектричних резонаторів. Знайдено загальні аналітичні рішення для частотних залежностей та амплітуд для одно-, дво- та три-вимірних решіток з різним розташуванням резонаторів.

Висновки. Розроблена теорія є основою для конструювання нових видів ліній затримки а також багатьох інших пристроїв оптичного діапазону довжин хвиль, які будуються на основі використання великого числа діелектричних резонаторів. Отримані нові аналітичні вирази для розрахунку параметрів хвиль, які розповсюджуються в складних структурах діелектричних резонаторів, дозволяють будувати нові більш ефективні математичні моделі різноманітних пристроїв оптичного зв’язку.

Ключові слова: *діелектричний резонатор; власні хвилі; решітка; зв'язана резонаторна лінія передачі; хвилевод; теорія збурень; лінія затримки.*

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