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OPTIMAL TIMING STRATEGIES IN BLOCKCHAIN BLOCK PROPOSALS BY ONE-BULLET SILENT DUELS WITH ONE-THIRD PROGRESSION

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Background. Silent duels and related timing games offer a surprisingly deep lens into certain core challenges in blockchain technology, especially when it comes to block proposal timing. Miners or validators effectively "compete" in a race to propose the next block. The success of a block proposal depends not only on when it happens but also on whether others have already succeeded or interfered — very much like the tension in a one-shot duel with uncertain outcomes. In block proposal timing for decentralized consensus protocols, a one-shot timing game models a blockchain setting, where participants (e. g., validators or miners) choose when to attempt block proposal or transaction insertion under uncertainty.

Objective. The paper aims to determine the best timing strategies for the participants. Considering two identical participants, the local objective is to find pure strategy solutions of a timing game (duel) with shooting uniform jitter.

Methods. A finite zero-sum game is considered, which models competitive interaction between two subjects to make the best discrete-time decision by limited observability. The moments to make a decision (to take an action, to shoot a bullet) are scheduled beforehand, and each of the subjects, alternatively referred to as the duelists, has a single bullet to shoot. Shooting is only possible during a standardized time span, where the bullet can be shot at only specified time moments. In the base pattern, apart from the duel beginning and final time moments, every following time moment is obtained by adding the third of the remaining span to the current moment. However, the precise time moment specification is not always realizable (e. g., due to the distance between neighbouring time moments being measured with finite accuracy) and so the internal moments are uniformly jittered. This means that they can be slightly shifted within the duel span. The duelist benefits from shooting as late as possible, but only when the duelist shoots first. Both the duelists act within the same conditions by linear shooting accuracy, and so the one-bullet silent duel is symmetric, regardless of the jitter. Therefore, its optimal value is 0 and the duelists have the same optimal strategies, although they still can be non-symmetric.

Results. By the one-third progression pattern with jitter, the 3×3 duel always has a pure strategy solution. The 4×4 duel is pure strategy solvable by any possible jitter except for jitter interval $\left(\left(\sqrt{19}-4\right)/9;1/6\right)$. Within this interval and interval $(-11/54;-1/18)$ the 5×5 duel is pure strategy non-solvable. The 6×6 duel is pure strategy solvable by any possible jitter except for jitter intervals $\left(\left(\sqrt{19}-4\right)/9;1/6\right)$ and $(-49/162;-1/18)$. Duels with seven to nine time moments are pure strategy solvable only by a jitter interval of $\left[-1/18;\left(\sqrt{19}-4\right)/9\right]$. Bigger $N\times N$ duels, having no fewer than 10 time moments, are pure strategy solvable only by a jitter interval of $\left[-1/18;2^{N-2}/3^{N-2}\right)$. The solutions for the one-third progression pattern are compared to the known solutions for the geometrical-progression pattern.

Conclusions. The duel pure strategy solutions obtained suggest a clear one-step-action strategic behaviour in progressive block proposal timing for decentralized consensus protocols under uncertainty of time slots to act. The main benefit is full fairness and a potential reward if the opponent acts non-optimally, even in a single proposal.

Keywords: *block proposal timing; one-bullet silent duel; linear accuracy; matrix game; pure strategy solution; progressing-by-one-third shooting moments.*

1. Progressive discrete silent duels with jitter

Silent duels and related timing games offer a surprisingly deep lens into certain core challenges in blockchain technology, especially when it comes to block proposal timing (e. g., in Proof-of-Work or Proof-of-Stake) [1], [2]. Miners or validators effectively "compete" in a race to propose the next block [3]. Each one makes a strategic timing decision — when to attempt a proposal, often based on stochastic processes. In Proof-of-Work, miners are waiting to "fire" their computational bullet by solving a puzzle first [4]. In Proof-of-Stake, validators might be selected with probabilities and have to choose when to propose or reveal information [5]. The success of a block proposal depends not only on when it happens but also on whether others have already succeeded or interfered — very much like the tension in a one-shot duel with uncertain outcomes [6], [7]. In block proposal timing for decentralized consensus protocols, a one-shot timing game models a blockchain setting, where participants (e. g., validators or miners) choose when to attempt block proposal or transaction insertion under uncertainty [1], [8], [9]. The general objective is to determine the best timing strategies for the participants.

A one-bullet discrete silent duel is a matrix game of timing [10], [11]

$$\langle X_N, Y_N, \mathbf{K}_N \rangle = \left\langle \{x_i\}_{i=1}^{N}, \{y_j\}_{j=1}^{N}, \mathbf{K}_N \right\rangle \quad (1)$$

with specified time moments as finite pure strategy sets

$$X_N = \{x_i\}_{i=1}^{N} = Y_N = \{y_j\}_{j=1}^{N} = T_N = \{t_q\}_{q=1}^{N} \subset [0;1]$$
$$\text{by } t_q < t_{q+1} \;\; \forall\, q = \overline{1, N-1}$$
$$\text{and } t_1 = 0,\; t_N = 1 \text{ for } N \in \mathbb{N} \setminus \{1\}, \quad (2)$$

in which the player, also referred to as the duelist, must make a decision (shoot one's single bullet) as late as possible but only to shoot first [7], [12]. Each of the duelists, allowed to shoot at any of N moments in (2), may not shoot until the final moment $t_N = 1$, but then the bullet is shot automatically anyway [8], [9]. The duelist is also featured with an accuracy function which is a non-decreasing function of time [7], [9], [12], [13].

Generally, in a Proof-of-Stake blockchain (like Ethereum post-merge, or Cardano, or Cosmos), the right to create the next block is randomly assigned to validators [4], [6]. These validators are selected based on how much cryptocurrency they have "staked" (locked up as collateral), giving them an economic incentive to follow the rules. Some of the validators are equal, and then they are both eligible to propose a block during a certain time window, which is broken into N short time slots (e. g., milliseconds or seconds). During each time slot a validator can choose to send a block proposal to the network — this is essentially a digital message containing a bundle of transactions, their signature, and some metadata (like block height). Once a block proposal is seen and accepted by the network, it becomes part of the blockchain — and only one proposal wins [1], [5], [6], [11]. Such a set-up (just like similar ones) is modelled by duel (1) with (2), where the validator is the duelist.

As both the duelists are presumed to have identical resources, the one-bullet silent duel is symmetric having a skew-symmetric payoff matrix (of rewards)

$$\mathbf{K}_N = \left[k_{ij}\right]_{N\times N} = \left[-k_{ji}\right]_{N\times N} = -\mathbf{K}_N^{\mathrm{T}}. \quad (3)$$

Therefore, its optimal value is 0 and the duelists have the same optimal strategies, although they still can be non-symmetric [7], [13]. For the case of linearized accuracy, the duelists' linear accuracy functions are

$$p_X(x) = x,\; p_Y(y) = y, \quad (4)$$

through which entry k_{ij} of payoff matrix (3) can be generally given as

$$k_{ij} = p_X(x_i) - p_Y(y_j) + p_X(x_i)\,p_Y(y_j)\,\mathrm{sign}(y_j - x_i) =$$
$$= x_i - y_j + x_i y_j\,\mathrm{sign}(y_j - x_i)$$
$$\text{for } i = \overline{1, N} \text{ and } j = \overline{1, N}. \quad (5)$$

Duel (1) with (2) — (5) is called progressive if its time moments, apart from, maybe, $t_N = 1$, are specified denser as the duel final approaches:

$$t_q - t_{q-1} > t_{q+1} - t_q \;\; \forall\, q = \overline{2, N-2}. \quad (6)$$

However, the precise specification is not always realizable (e. g., due to the distance between neighbouring time moments is measured with finite accuracy) and so internal moments $\{t_q\}_{q=2}^{N-1}$ are uniformly jittered [14], [15]. This means that moments $\{t_q\}_{q=2}^{N-1}$ can be slightly shifted within duel span $[0;1]$ by still obeying (6). In block proposal timing for decentralized consensus protocols, the jitter models uncertainty of time slots to act.

2. Geometrical progression pattern

The particular case of duel (1) with (2) — (6) was considered in [14], where

$$t_q = \xi + \sum_{l=1}^{q-1} 2^{-l} = \xi + \frac{2^{q-1}-1}{2^{q-1}}$$
$$\text{for } q = \overline{2, N-1} \text{ and } \xi \in \left(-\frac{1}{2}; \frac{1}{2^{N-2}}\right). \quad (7)$$

Time moments $\{t_q\}_{q=2}^{N-1}$ specified by (7) is a shooting uniform jitter, which slightly moves [12], [15] points $\{t_q\}_{q=2}^{N-1}$ by [14]

$$t_q = t_{q-1} + \frac{1-t_{q-1}}{2} = \frac{1+t_{q-1}}{2} = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1}-1}{2^{q-1}}$$
$$\text{for } q = \overline{2, N-1} \quad (8)$$

within the duel span $[0;1]$ not violating their relative order (topology) within interval $[t_2; t_{N-1}]$ by still obeying (6). The case with $\xi > 0$ is called a positive jitter, and the case with $\xi < 0$ is called a negative jitter. Time moment

$$t_q = \xi + \frac{2^{q-1}-1}{2^{q-1}} \text{ at } q \in \left\{\overline{2, N-1}\right\}$$

is called positively ξ-jittered moment and negatively

$|\xi|$-jittered moment by $\xi > 0$ and $\xi < 0$, respectively. Inasmuch as $t_2 = \frac{1}{2}$, the negative jitter must be higher than $-\frac{1}{2}$ to ensure

$$t_2 = \xi + \frac{1}{2} > t_1 = 0 .$$

Inasmuch as

$$t_N - t_{N-1} = 1 - \frac{2^{N-2} - 1}{2^{N-2}} = \frac{1}{2^{N-2}} , \tag{9}$$

the positive jitter must be lower than (9) to ensure

$$t_N = 1 > \xi + \frac{2^{N-2} - 1}{2^{N-2}} .$$

For the case of $\xi = 0$, the pure strategy solutions of duel (1) with (2) — (6) and (8) are studied in [13]. For a trivial 3×3 duel, in which the duelist possesses just one moment of possible shooting between the duel beginning and final moments, any pure strategy situation, not containing the duel beginning moment, is optimal. In 3×3 duels and bigger the pure strategy solution is situation

$$\{x_2, y_2\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\} . \tag{10}$$

In duels bigger than the 3×3 duel, optimal pure strategy situation (10) is the single one.

The case of jitter for this duel, i. e. when $\xi \neq 0$ in (7), is reasonably split into subcases of positive and negative jitter. Thus, in the duel with a positive jitter, the only optimal behaviour of the duelist is to shoot at the positively ξ-jittered middle $t_2 = \xi + \frac{1}{2}$ of the duel span.

3. One-third progression pattern

Due to geometrical progression pattern (8) tightens time moments too dense, it lacks a reasonable final-to-penultimate ratio

$$\frac{t_N}{t_{N-1}} = \frac{1}{t_{N-1}} , \tag{11}$$

which is

$$\frac{1}{t_{N-1}} = \frac{2^{N-2}}{2^{N-2} - 1} . \tag{12}$$

Hence, another progression pattern is to be considered. According to this pattern, every next time moment is obtained by adding the third of the remaining span to the current moment:

$$t_q = t_{q-1} + \frac{1 - t_{q-1}}{3} = \frac{1 + 2t_{q-1}}{3} = \sum_{l=1}^{q-1} \frac{2^{l-1}}{3^l} = \frac{3^{q-1} - 2^{q-1}}{3^{q-1}}$$
$$\text{for } q = \overline{2, N-1} . \tag{13}$$

For one-third progression pattern (13), final-to-penultimate ratio (11) is

$$\frac{1}{t_{N-1}} = \frac{3^{N-2}}{3^{N-2} - 2^{N-2}} . \tag{14}$$

The difference between final-to-penultimate ratios (14) and (12) is

$$\frac{3^{N-2}}{3^{N-2} - 2^{N-2}} - \frac{2^{N-2}}{2^{N-2} - 1} =$$
$$= \frac{3^{N-2} \cdot 2^{N-2} - 3^{N-2} - 3^{N-2} \cdot 2^{N-2} + 2^{N-2} \cdot 2^{N-2}}{\left(3^{N-2} - 2^{N-2}\right)\left(2^{N-2} - 1\right)} =$$
$$= \frac{2^{N-2} \cdot 2^{N-2} - 3^{N-2}}{\left(3^{N-2} - 2^{N-2}\right)\left(2^{N-2} - 1\right)} =$$
$$= \frac{4^{N-2} - 3^{N-2}}{\left(3^{N-2} - 2^{N-2}\right)\left(2^{N-2} - 1\right)} > 0 \text{ for } N \in \mathbb{N} \setminus \{1, 2\} . \tag{15}$$

Difference (15) shows that one-third progression pattern (13) tightens time moments less dense leaving for the duelist a longer gap between the penultimate and final moment of possible shooting.

Inasmuch as $t_2 = \frac{1}{3}$ by one-third progression pattern (13), its negative jitter must be higher than $-\frac{1}{3}$ to ensure

$$t_2 = \xi + \frac{1}{3} > t_1 = 0 .$$

Inasmuch as

$$t_N - t_{N-1} = 1 - \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} = \frac{2^{N-2}}{3^{N-2}} , \tag{16}$$

the positive jitter must be lower than difference (16) to ensure

$$t_N = 1 > \xi + \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} .$$

Herein, the local objective is to find pure strategy solutions of duel (1) with (2) — (6) and shooting uniform jitter

$$t_q = \xi + \frac{3^{q-1} - 2^{q-1}}{3^{q-1}}$$

$$\text{for } q = \overline{2, N-1} \text{ and } \xi \in \left(-\frac{1}{3}; \frac{2^{N-2}}{3^{N-2}} \right) \quad (17)$$

for $N \in \mathbb{N} \setminus \{1, 2\}$.

4. Saddle points in matrix (3)

Inasmuch as a pure strategy solution of duel (1) with skew-symmetric payoff matrix (3) corresponds to a saddle point of this matrix having entries (5), only a zero entry of matrix (3) can be a saddle point [7], [16]. Therefore, a row containing a negative entry does not contain saddle points; neither does the respective column containing the positive entry. Hence, it is conventionally possible to conclude only on saddle points in definite rows of matrix (3), which imply the same conclusions on saddle points in respective columns. Thus, a nonnegative row of matrix (3) with entries (5) contains a saddle point on the main diagonal of the matrix [7]. If a row contains only positive entries, except for the main diagonal entry, all the other $N-1$ rows of the respective column contain negative entries, and thus this row contains a single saddle point which is the single one in the duel. Moreover, inasmuch as

$$k_{1j} = -y_j < 0 \quad \forall j = \overline{2, N}$$

then the first row of matrix (3) with entries (5) is not an optimal strategy of the first duelist, and thus situation

$$\{x_1, y_1\} = \{0, 0\}$$

is never optimal in the duel [7], [9], [13], [14]. Next, if row i^* contains entry

$$k_{i^* j_*} = 0 \text{ by } j_* \in \left\{ \overline{2, N} \right\} \text{ and } i^* \neq j_*,$$

where situation

$$\left\{ x_{i^*}, y_{j_*} \right\} = \left\{ t_{i^*}, t_{j_*} \right\} \quad (18)$$

is optimal, then the duel has at least four pure strategy solutions [7]: symmetric situations

$$\left\{ x_{i^*}, y_{i^*} \right\} = \left\{ t_{i^*}, t_{i^*} \right\}, \quad (19)$$

$$\left\{ x_{j_*}, y_{j_*} \right\} = \left\{ t_{j_*}, t_{j_*} \right\}, \quad (20)$$

and non-symmetric situations (18),

$$\left\{ x_{j_*}, y_{i^*} \right\} = \left\{ t_{j_*}, t_{i^*} \right\}. \quad (21)$$

It is noteworthy that, despite in duels (1) with (3) the optimal behaviour of the duelists is the same, optimal situations (18) and (21) are, in fact, non-symmetric. Nevertheless, symmetry in optimal situations (19) and (20) and non-symmetry in optimal situations (18) and (21) are equivalent owing to the duelist's optimal strategies in those solutions are interchangeable without affecting the duel outcome.

5. Three moments to shoot

The most trivial geometrical progression pattern by (7) is a triple

$$T_3 = \{t_1, t_2, t_3\} = \left\{ 0, \frac{1}{2} + \xi, 1 \right\} \text{ for } \xi \in \left(-\frac{1}{2}; \frac{1}{2} \right). \quad (22)$$

The respective 3×3 duel with a positive jitter has a single optimal solution, by which the best decision is to shoot at the positively ξ-jittered middle $t_2 = \frac{1}{2} + \xi$ of the duel span. In the case of a negative jitter, the optimal behaviour of the duelist is to shoot at the final moment. Does the solution keep this structure for the 3×3 duel with one-third progression pattern

$$T_3 = \{t_1, t_2, t_3\} = \left\{ 0, \frac{1}{3} + \xi, 1 \right\} \text{ for } \xi \in \left(-\frac{1}{3}; \frac{2}{3} \right) \quad (23)$$

or not? The answer follows.

Theorem 1. Duel (1) by (3), (5) and (23)

$$\langle X_3, Y_3, \mathbf{K}_3 \rangle = \left\langle \left\{ 0, \frac{1}{3} + \xi, 1 \right\}, \left\{ 0, \frac{1}{3} + \xi, 1 \right\}, \mathbf{K}_3 \right\rangle \quad (24)$$

has a single optimal situation

$$\{x_2, y_2\} = \left\{ \frac{1}{3} + \xi, \frac{1}{3} + \xi \right\} \quad (25)$$

by

$$\xi \in \left(\frac{1}{6}; \frac{2}{3} \right). \quad (26)$$

By $\xi = \frac{1}{6}$ duel (24) has four optimal pure strategy situations (25),

$$\{x_2, y_3\} = \left\{ \frac{1}{3} + \xi, 1 \right\}, \quad (27)$$

$$\{x_3, y_2\} = \left\{ 1, \frac{1}{3} + \xi \right\}, \quad (28)$$

$$\{x_3, y_3\} = \{1, 1\}. \quad (29)$$

By

$$\xi \in \left(-\frac{1}{3}; \frac{1}{6} \right) \quad (30)$$

duel (24) has a single optimal situation (29).

Proof. Upon plugging elements of (23) into (5) for (2) and $N = 3$, the respective payoff matrix is

$$\mathbf{K}_3 = \left[k_{ij}\right]_{3\times 3} = \begin{bmatrix} 0 & -\frac{1}{3}-\xi & -1 \\ \frac{1}{3}+\xi & 0 & -\frac{1}{3}+2\xi \\ 1 & \frac{1}{3}-2\xi & 0 \end{bmatrix}. \quad (31)$$

Situation (25) is single optimal if inequalities

$$k_{21} = \frac{1}{3}+\xi > 0 \quad (32)$$

and

$$k_{23} = -\frac{1}{3}+2\xi > 0 \quad (33)$$

hold simultaneously, whence

$$\xi > -\frac{1}{3} \text{ and } \xi > \frac{1}{6} \quad (34)$$

and inequalities (34) with (23) imply condition (26). If $\xi = \frac{1}{6}$ then

$$k_{22} = k_{23} = k_{32} = k_{33} = 0,$$

while inequality (32) still holds and $k_{31} = 1$. Therefore, matrix (31) by $\xi = \frac{1}{6}$ has four saddle points: (25), (27) — (29). By condition (30) inequality

$$k_{32} = \frac{1}{3} - 2\xi > 0 \quad (35)$$

holds and situation (29) is the single saddle point. □

So, Theorem 1 reveals the boundary value of jitter, which is $\xi = \frac{1}{6}$, that separates the two solution cases in 3×3 duels with one-third progression pattern (23). The 3×3 duel with geometrical progression pattern (22) does not have such a boundary value (the case of $\xi = 0$ is classified separately).

6. Second moment optimality

In duel (1) with (2) — (6) and shooting uniform jitter (7) for $\xi > 0$, the only best decision is to shoot at the positively ξ-jittered middle $t_2 = \frac{1}{2}+\xi$ of the duel span, where the number of time moments is three or more [14]. For one-third progression pattern with jitter (17) for three time moments, this case is narrowed to the half-unit-length interval $\left(\frac{1}{6}; \frac{2}{3}\right)$, by which the positively ξ-jittered middle $t_2 = \frac{1}{3}+\xi$ is single optimal (Theorem 1). See right below, whether this moment remains optimal for bigger duels.

Theorem 2. In duel (1) with (2) — (6) and (17) by no fewer than four moments to shoot, situation (25) is single optimal when only

$$\xi \in \left[\frac{1}{6}; \frac{2^{N-2}}{3^{N-2}}\right) \quad (36)$$

and the duel has four to six moments to shoot.

Proof. Situation (25) is single optimal if the second row of matrix (3) is positive except for entry k_{22}. In this row, inequality (32) holds and

$$k_{2j} = \frac{1}{3}+\xi - y_j + \left(\frac{1}{3}+\xi\right)\cdot y_j = \\ = \frac{1}{3}+\xi - \left(\frac{2}{3}-\xi\right)\cdot y_j \quad \forall\, j = \overline{3, N}. \quad (37)$$

Due to

$$\frac{2}{3}-\xi > 0 \quad \forall \xi \in \left(-\frac{1}{3}; \frac{2^{N-2}}{3^{N-2}}\right) \text{ for } N \in \mathbb{N} \setminus \{1, 2, 3\}, \quad (38)$$

entry (37) is a decreasing function of y_j. Entry

$$k_{2N} = \frac{1}{3}+\xi - 1 + \frac{1}{3}+\xi = 2\xi - \frac{1}{3} > 0 \quad (39)$$

if (34) is true. However, inequality (39) is possible if interval

$$\left(\frac{1}{6}; \frac{2^{N-2}}{3^{N-2}}\right) \text{ for } N \in \mathbb{N} \setminus \{1, 2, 3\} \quad (40)$$

is nonempty, i. e.

$$\frac{2^{N-2}}{3^{N-2}} - \frac{1}{6} = \frac{2^{N-1} - 3^{N-3}}{2\cdot 3^{N-2}} > 0. \quad (41)$$

Consider function

$$f(N) = 2^{N-1} - 3^{N-3} \quad (42)$$

in the numerator of the fraction in (41). The first derivative of function (42) is

$$\frac{df}{dN} = 2^{N-1}\cdot \ln 2 - 3^{N-3}\cdot \ln 3. \quad (43)$$

First derivative (43) turns into zero if

$$2^{N-1} \cdot \ln 2 = 3^{N-3} \cdot \ln 3, \tag{44}$$

whence

$$\frac{2^{N-1}}{3^{N-3}} = \frac{\ln 3}{\ln 2},$$

$$\left(\frac{2}{3}\right)^{N-1} = \frac{\ln 3}{9 \ln 2},$$

$$\ln\left(\frac{2}{3}\right)^{N-1} = \ln \frac{\ln 3}{9 \ln 2},$$

$$(N-1)(\ln 2 - \ln 3) = \ln(\ln 3) - \ln(9 \ln 2),$$

$$N(\ln 2 - \ln 3) - \ln 2 + \ln 3 = \ln(\ln 3) - 2\ln 3 - \ln(\ln 2),$$

$$N = \frac{\ln(\ln 3) - 3\ln 3 - \ln(\ln 2) + \ln 2}{\ln 2 - \ln 3}. \tag{45}$$

Value (45) is such that

$$5.28 < \frac{\ln(\ln 3) - 3\ln 3 - \ln(\ln 2) + \ln 2}{\ln 2 - \ln 3} < 5.29,$$

so function (42) has the single extremum. The second derivative of function (42) is

$$\frac{d^2 f}{dN^2} = 2^{N-1} \cdot (\ln 2)^2 - 3^{N-3} \cdot (\ln 3)^2. \tag{46}$$

Second derivative (46) at point (45) is negative, so (45) is the single maximum point. As

$$f(4) = 5,\ f(6) = 5,\ f(7) = -17,$$

then function (42) is negative for $N \geqslant 7$ while it is positive for $N \in \{4, 5, 6\}$. Therefore, inequality (41) for integer N above 3 holds only for $N \in \{4, 5, 6\}$. Under only this condition the interval in (40) is nonempty making the second row of matrix (3) positive except for entry k_{22}, i. e. when situation (25) is the single solution in the duel having four to six moments to shoot by

$$\xi \in \left(\frac{1}{6}; \frac{2^{N-2}}{3^{N-2}}\right). \tag{47}$$

When $\xi = \frac{1}{6}$, entries $k_{22} = k_{2N} = 0$ but the remaining $N-2$ entries in the second row are positive. In the last row, entry

$$k_{N,N-1} = 1 - 2 \cdot \frac{1}{6} - 2 \cdot \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} = \frac{2}{3} - 2 \cdot \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} < 0 \tag{48}$$

due to

$$\frac{2^{N-2}}{3^{N-2}} < \frac{2}{3} = 1 - \frac{1}{3} \text{ for } N \geqslant 4,$$

$$\frac{1}{3} < 1 - \frac{2^{N-2}}{3^{N-2}} = \frac{3^{N-2} - 2^{N-2}}{3^{N-2}},$$

$$\frac{2}{3} < 2 \cdot \frac{3^{N-2} - 2^{N-2}}{3^{N-2}},$$

and thus this row cannot be an optimal strategy. This finalizes proving the singleness of solution (25) by (36) for $N \in \{4, 5, 6\}$. $\square$

In fact, Theorem 2 implies that the second moment is never optimal in duels having no fewer than seven moments to shoot. The question of whether the third moment is optimal in such duels is answered right below.

7. Third moment optimality

In duel (1) with (2) — (6) and shooting uniform jitter (7) for no fewer than four time moments to shoot, the third ξ-jittered moment $t_3 = \frac{3}{4} + \xi$ is single optimal if [14]

$$\xi \in \left(-\frac{1}{4}; \frac{\sqrt{17} - 5}{8}\right].$$

See right below, whether the third ξ-jittered moment $t_3 = \frac{5}{9} + \xi$ by the one-third progression pattern can be optimal.

Theorem 3. In duel (1) with (2) — (6) and (17), situation

$$\{x_3, y_3\} = \left\{\frac{5}{9} + \xi, \frac{5}{9} + \xi\right\} \tag{49}$$

is single optimal when only

$$\xi \in \left[-\frac{1}{18}; \frac{\sqrt{19} - 4}{9}\right] \tag{50}$$

for five to nine time moments and

$$\xi \in \left[-\frac{1}{18}; \frac{2^{N-2}}{3^{N-2}}\right) \tag{51}$$

for duels having no fewer than 10 time moments. Situation (49) is single optimal in the 4×4 duel when only

$$\xi \in \left(-\frac{1}{18}; \frac{\sqrt{19} - 4}{9}\right]. \tag{52}$$

At

$$\xi = -\frac{1}{18} \tag{53}$$

the 4×4 duel has four optimal pure strategy solutions: situations (49),

$$\{x_3, y_4\} = \left\{\frac{5}{9} + \xi, 1\right\}, \tag{54}$$

$$\{x_4, y_3\} = \left\{1, \frac{5}{9} + \xi\right\}, \tag{55}$$

$$\{x_4, y_4\} = \{1, 1\}. \tag{56}$$

Proof. In the third row of matrix (3) entry

$$k_{31} = \frac{5}{9} + \xi > 0 \tag{57}$$

and entry

$$k_{32} = \frac{5}{9} + \xi - \frac{1}{3} - \xi - \left(\frac{5}{9} + \xi\right)\left(\frac{1}{3} + \xi\right) = \\ = -\xi^2 - \frac{8}{9}\xi + \frac{1}{27} \geqslant 0 \tag{58}$$

if

$$\xi \in \left[-\frac{\sqrt{19}+4}{9}; \frac{\sqrt{19}-4}{9}\right]. \tag{59}$$

Entry

$$k_{3j} = \frac{5}{9} + \xi - y_j + \left(\frac{5}{9} + \xi\right)\cdot y_j = \\ = \frac{5}{9} + \xi - \left(\frac{4}{9} - \xi\right)\cdot y_j \quad \forall\, j = \overline{4, N}. \tag{60}$$

Due to

$$\frac{4}{9} - \xi > 0 \quad \forall \xi \in \left(-\frac{1}{3}; \frac{2^{N-2}}{3^{N-2}}\right) \\ \text{for } N \in \mathbb{N} \setminus \{1, 2, 3\}, \tag{61}$$

entry (60) is a decreasing function of y_j. Entry

$$k_{3N} = \frac{5}{9} + \xi - 1 + \frac{5}{9} + \xi = 2\xi + \frac{1}{9} \geqslant 0 \tag{62}$$

if

$$\xi \geqslant -\frac{1}{18}, \tag{63}$$

where

$$-\frac{\sqrt{19}+4}{9} < -\frac{1}{3} < -\frac{1}{18} < 0 < \frac{\sqrt{19}-4}{9} < \frac{1}{6}. \tag{64}$$

So, situation (49) is optimal if (59) and (63) are true, which via (64) leads to condition (50). However, it is easily checked that

$$\frac{\sqrt{19}-4}{9} < \frac{2^{N-2}}{3^{N-2}} \text{ for } N \in \{5, 6, 7, 8, 9\} \tag{65}$$

and

$$\frac{2^{N-2}}{3^{N-2}} < \frac{\sqrt{19}-4}{9} \text{ for } N \geqslant 10. \tag{66}$$

Inequality (65) means that situation (49) is optimal in 5×5 to 9×9 duels if (50) is true, and inequality (66) means that situation (49) is optimal in duels bigger than the 9×9 duel if (51) is true.

As (64) holds, then, as a corollary from Theorem 2, situation (25) is not optimal by (50) for $N \in \{5, 6, 7, 8, 9\}$ and it is not optimal by (51) for $N \in \mathbb{N} \setminus \{\overline{1, 9}\}$. This resolves the case of when $k_{32} = 0$ by $\xi = \frac{\sqrt{19}-4}{9}$, while the remaining $N-2$ entries in the third row are positive. Another case to resolve is when $k_{3N} = 0$ by (53), while the remaining $N-2$ entries in the third row are positive. In the last row, entry

$$k_{N,N-1} = 1 + 2\cdot\frac{1}{18} - 2\cdot\frac{3^{N-2} - 2^{N-2}}{3^{N-2}} = \\ = \frac{10}{9} - 2\cdot\frac{3^{N-2} - 2^{N-2}}{3^{N-2}} < 0 \tag{67}$$

due to

$$\frac{2^{N-2}}{3^{N-2}} < \frac{4}{9} = 1 - \frac{5}{9} \text{ for } N \geqslant 5,$$

$$\frac{5}{9} < 1 - \frac{2^{N-2}}{3^{N-2}} = \frac{3^{N-2} - 2^{N-2}}{3^{N-2}},$$

$$\frac{10}{9} < 2\cdot\frac{3^{N-2} - 2^{N-2}}{3^{N-2}},$$

whence the last row cannot be an optimal strategy by (53) for no fewer than five moments to shoot. Hence, optimal situation (49) is single for 5×5 and bigger duels.

In the 4×4 duel, situation (25) is not optimal by (50). Owing to (57) and the reasoning by statements (58) — (64), which support the case with four time moments as well, optimal situation (49) is single by (52). If (53) is true, then

$$k_{33} = k_{34} = k_{43} = k_{44} = 0,$$

and

$$k_{31} = \frac{1}{2},$$

$$k_{32} = \frac{1}{2} - \frac{5}{18} - \frac{1}{2} \cdot \frac{5}{18} = \frac{1}{12},$$

$$k_{41} = 1,$$

$$k_{42} = 1 - 2 \cdot \frac{5}{18} = \frac{4}{9},$$

which implies optimality of the quadruple of situations (49), (54) — (56). □

A corollary from Theorem 3 is that the third moment is not optimal in duels having no fewer than four moments if

$$\xi \in \left(-\frac{1}{3}; -\frac{1}{18} \right). \tag{68}$$

It is natural to continue investigation beyond the third moment for such duels whose moments are jittered below the value of (53).

8. Beyond the third moment

First consider fourth moment $t_4 = \frac{19}{27} + \xi$ that follows moment $t_3 = \frac{5}{9} + \xi$.

Theorem 4. In duel (1) with (2) — (6) and (17) by no fewer than five moments to shoot, the fourth moment is never optimal, whichever jitter is.

Proof. Situation

$$\{x_4, y_4\} = \left\{ \frac{19}{27} + \xi, \frac{19}{27} + \xi \right\} \tag{69}$$

can be optimal only if

$$k_{43} = \frac{19}{27} + \xi - \frac{5}{9} - \xi - \left(\frac{19}{27} + \xi \right)\left(\frac{5}{9} + \xi \right) = \\ = -\xi^2 - \frac{34}{27}\xi - \frac{59}{243} \geqslant 0. \tag{70}$$

Inequality (70) holds if

$$\xi \in \left[-\frac{4\sqrt{7}+17}{27}; \frac{4\sqrt{7}-17}{27} \right]. \tag{71}$$

Besides, situation (69) can be optimal only if

$$k_{4N} = \frac{19}{27} + \xi - 1 + \frac{19}{27} + \xi = 2\xi + \frac{11}{27} \geqslant 0. \tag{72}$$

Inequality (72) holds if

$$\xi \geqslant -\frac{11}{54}, \tag{73}$$

where

$$-\frac{1}{3} < \frac{4\sqrt{7}-17}{27} < -\frac{11}{54}. \tag{74}$$

Inequality (74) implies that situation (69) is never optimal because inequality (73) cannot hold when (71) is true and vice versa. □

Now, before considering next time moments, it will be needful to use the following lemma.

Lemma 1. Entry

$$k_{i+1,i} = x_{i+1} - y_i - x_{i+1}y_i \tag{75}$$

of payoff matrix (3) as a function of $i = \overline{2, N-2}$ is a decreasing function.

Proof. In entry (75),

$$x_i = y_i = \xi + \frac{3^{i-1} - 2^{i-1}}{3^{i-1}}$$

and

$$x_{i+1} = y_{i+1} = \xi + \frac{3^i - 2^i}{3^i},$$

so

$$x_{i+1} - y_i = \\ = \xi + \frac{3^i - 2^i}{3^i} - \xi - \frac{3^{i-1} - 2^{i-1}}{3^{i-1}} = \frac{2^{i-1}}{3^{i-1}} - \frac{2^i}{3^i}. \tag{76}$$

Denote the difference in (76) by

$$a = \frac{2^{i-1}}{3^{i-1}} - \frac{2^i}{3^i}. \tag{77}$$

Then

$$x_{i+1} = y_{i+1} = x_i + a. \tag{78}$$

Similarly to (78), value (77) is used to re-write entry

$$k_{i+2,i+1} = x_{i+2} - y_{i+1} - x_{i+2}y_{i+1}, \tag{79}$$

where

$$x_{i+2} - y_{i+1} = \\ = \xi + \frac{3^{i+1} - 2^{i+1}}{3^{i+1}} - \xi - \frac{3^i - 2^i}{3^i} = \frac{2^i}{3^i} - \frac{2^{i+1}}{3^{i+1}} = \frac{2}{3}a \tag{80}$$

and

$$x_{i+2} = x_{i+1} + \frac{2}{3}a = x_i + a + \frac{2}{3}a = x_i + \frac{5}{3}a. \tag{81}$$

Simplify the difference between entry (75) and entry (79) for $i=\overline{2, N-3}$ by using (76) — (78), (80), (81) to express it with only x_i and (77):

$$\begin{gathered}
k_{i+1,i}-k_{i+2,i+1}= \\
=x_{i+1}-y_i-x_{i+1}y_i-\left(x_{i+2}-y_{i+1}-x_{i+2}y_{i+1}\right)= \\
=a-\left(x_i+a\right)x_i-\left(\frac{2}{3}a-\left(x_i+\frac{5}{3}a\right)\left(x_i+a\right)\right)= \\
=a-x_i^2-ax_i-\frac{2}{3}a+x_i^2+\frac{5}{3}ax_i+ax_i+\frac{5}{3}a^2= \\
=\frac{1}{3}a+\frac{5}{3}ax_i+\frac{5}{3}a^2= \\
=\frac{a}{3}\left(5a+5x_i+1\right)>0 \text{ for } i=\overline{2, N-3}.
\end{gathered} \tag{82}$$

Inequality in the last line of (82) directly means that entry (75) is a decreasing function of $i=\overline{2, N-2}$. □

Now, it is just time to proceed with later moments, moving onward by considering 5×5 and bigger duels.

Theorem 5. In duel (1) with (2) — (6) and (17) by no fewer than five moments to shoot, the time moments between the fourth and the $(N-1)$-th one are never optimal, whichever jitter is.

Proof. The assertion claims that time moments

$$t_q=\xi+\frac{3^{q-1}-2^{q-1}}{3^{q-1}} \text{ for } q=\overline{4, N-1}$$
$$\text{and } \xi\in\left(-\frac{1}{3};\frac{2^{N-2}}{3^{N-2}}\right) \text{ by } N\in\mathbb{N}\setminus\left\{\overline{1, 4}\right\} \tag{83}$$

in 5×5 and bigger duels are never optimal. The non-optimality of the fourth moment is proved by Theorem 4. To prove the non-optimality of time moments

$$\left\{t_q\right\}_{q=5}^{N-1} \text{ for } N\in\mathbb{N}\setminus\left\{\overline{1, 5}\right\},$$

consider the fifth row of matrix (3) and its entry k_{54}:

$$\begin{gathered}
k_{54}=\frac{65}{81}+\xi-\frac{19}{27}-\xi-\left(\frac{65}{81}+\xi\right)\left(\frac{19}{27}+\xi\right)= \\
=-\xi^2-\frac{122}{81}\xi-\frac{1019}{2187}\geqslant 0
\end{gathered} \tag{84}$$

if

$$\xi\in\left[-\frac{2\sqrt{166}+61}{81};\frac{2\sqrt{166}-61}{81}\right], \tag{85}$$

where

$$\frac{2\sqrt{166}-61}{81}<-\frac{1}{3}. \tag{86}$$

Inequality (86) means that inequality (84) is impossible due to $\xi>-\frac{1}{3}$. Therefore, $k_{54}<0$. Inasmuch as entry (75) is a decreasing function of $i=\overline{2, N-2}$ (Lemma 1), entries

$$k_{i+1,i}<0 \text{ for } i=\overline{4, N-2}. \tag{87}$$

Inequalities (87) along with Theorem 4 directly imply that the rows from the fourth down to the $(N-1)$-th one do not contain saddle points. □

In 4×4 duels, the third moment is followed only by the final moment.

Theorem 6. In 4×4 duel (1) with (2) — (6) and (17) for $N=4$ and (68) situation (56) is single optimal.

Proof. Situation (56) is single optimal if the last, fourth, row of matrix (3) is positive, except for entry $k_{44}=0$. Entry

$$k_{4j}=1-2y_j>0 \text{ for } j=\overline{1, 3}. \tag{88}$$

Inasmuch as

$$y_1<y_2<y_3=\frac{5}{9}+\xi,$$

inequality (88) holds if

$$\frac{5}{9}+\xi<\frac{1}{2},$$

whence

$$\xi<-\frac{1}{18},$$

i. e. by (68) the final moment is the only optimal strategy. □

Amazingly enough, but Theorem 5 also holds for duels (1) with (2) — (6) and shooting uniform jitter (7) [14]. In such duels, having no fewer than five moments to shoot, the final moment is single optimal if

$$\xi\in\left(-\frac{1}{2};-\frac{1}{2}+\frac{1}{2^{N-2}}\right]. \tag{89}$$

See right below whether at sufficiently high negative jitter the final moment is single optimal by the one-third progression pattern in 5×5 and bigger duels.

9. Final moment

Theorem 7. In 5×5 duel (1) with (2) — (6) and

(17) for $N=5$ situation

$$\{x_N, y_N\} = \{1, 1\} \tag{90}$$

is single optimal by

$$\xi \in \left(-\frac{1}{3}; -\frac{11}{54}\right]. \tag{91}$$

In 6×6 duel (1) with (2) — (6) and (17) for $N=6$ situation (90) is single optimal by

$$\xi \in \left(-\frac{1}{3}; -\frac{49}{162}\right]. \tag{92}$$

Duels having no fewer than seven moments are not solved in pure strategies by (68).

Proof. Situation (90) is optimal if entry

$$k_{Nj} = 1 - 2y_j \geqslant 0 \text{ for } j = \overline{1, N-1}. \tag{93}$$

Obviously, entry (93) is a strictly decreasing function of y_j and thus, in the last row of matrix (3), entry $k_{N,N-1}$ is minimal not considering entry $k_{N,N}=0$. Inasmuch as

$$y_j < y_{N-1} = \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} + \xi \text{ for } j = \overline{1, N-2},$$

inequality (93) holds if

$$\frac{3^{N-2} - 2^{N-2}}{3^{N-2}} + \xi \leqslant \frac{1}{2},$$

whence

$$\xi \leqslant \frac{1}{2} - \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} = -\frac{1}{2} + \frac{2^{N-2}}{3^{N-2}}. \tag{94}$$

Clearly, value

$$-\frac{1}{2} + \frac{2^{N-2}}{3^{N-2}}$$

is a decreasing function of natural N, so by $N=5$ inequality (94) turns into inequality

$$\xi \leqslant -\frac{1}{2} + \frac{8}{27} = -\frac{11}{54} \tag{95}$$

and by $N=6$ inequality (94) turns into inequality

$$\xi \leqslant -\frac{1}{2} + \frac{16}{81} = -\frac{49}{162}, \tag{96}$$

whereas by $N \geqslant 7$ inequality (94) turns into inequality

$$\xi \leqslant -\frac{1}{2} + \frac{32}{243} = -\frac{179}{486} < -\frac{1}{3}. \tag{97}$$

As inequality

$$\xi < -\frac{1}{3}$$

is impossible, inequality (97) means that inequality (93) is impossible for duels having no fewer than seven moments to shoot. Along with Theorem 3, this also implies that duels having no fewer than seven moments are not solved in pure strategies by (68).

Meanwhile,

$$-\frac{1}{3} < -\frac{11}{54} < -\frac{1}{18}. \tag{98}$$

Then inequality (98) means that by (91) the final moment is optimal in the 5×5 duel, in which entry

$$k_{N,N-1} = k_{54} = 0$$

if only

$$\xi = -\frac{11}{54}. \tag{99}$$

However, at (99) entry

$$k_{N-1,N-2} = k_{43}$$

being the parabola in inequality (70) is negative due to inequality (70) holds by (71), where the latter is not true by (99) as (74) holds. Therefore, at (99) situation (90) for $N=5$ remains single optimal.

Another inequality is

$$-\frac{1}{3} < -\frac{49}{162} < -\frac{11}{54} < -\frac{1}{18}. \tag{100}$$

Inequality (100) means that by (92) the final moment is optimal in the 6×6 duel, in which entry

$$k_{N,N-1} = k_{65} = 0$$

if only

$$\xi = -\frac{49}{162}. \tag{101}$$

However, at (101) entry

$$k_{N-1,N-2} = k_{54}$$

being the parabola in inequality (84) is negative due to inequality (84) holds by (85), where the latter is not true by (101) as (86) holds. Therefore, at (101) situation (90) for $N=6$ remains single optimal. □

10. Non-solvability in pure strategies

By one-third progression pattern with jitter (17), 3×3 duels are pure strategy solvable at any jitter, but the 4×4 duel is not solved in pure strategies if

$$\xi \in \left(\frac{\sqrt{19}-4}{9}; \frac{1}{6} \right). \tag{102}$$

This is a corollary from Theorems 2, 3, 6. The 5×5 duel is not solved in pure strategies by (102) and by

$$\xi \in \left(-\frac{11}{54}; -\frac{1}{18} \right). \tag{103}$$

This is a corollary from Theorems 2, 3, 4, 5, 7. The same assertions are followed by a corollary that the 6×6 duel is not solved in pure strategies by (102) and by

$$\xi \in \left(-\frac{49}{162}; -\frac{1}{18} \right). \tag{104}$$

Duels with seven to nine time moments are pure strategy non-solvable by

$$\xi \in \left(\frac{\sqrt{19}-4}{9}; \frac{2^{N-2}}{3^{N-2}} \right) \text{ for } N \in \{7, 8, 9\} \tag{105}$$

and by (68), i. e. 7×7, 8×8, 9×9 duels are not solved in pure strategies if the positive jitter is higher than value $\frac{\sqrt{19}-4}{9}$ or the negative jitter is lower than value $-\frac{1}{18}$ (Theorems 2, 3, 4, 5, 7). Non-solvability in pure strategies is recognized easier in bigger duels (having no fewer than 10 time moments): they are pure strategy non-solvable when just the negative jitter is lower than value $-\frac{1}{18}$.

11. Conclusion

By one-third progression pattern with jitter (17), the 3×3 duel always has a pure strategy solution (Theorem 1). The 4×4 duel is pure strategy solvable by any possible jitter

$$\xi \in \left\{ \left(-\frac{1}{3}; \frac{4}{9} \right) \setminus \left(\frac{\sqrt{19}-4}{9}; \frac{1}{6} \right) \right\}. \tag{106}$$

In the 4×4 duel any moment, except for the duel beginning moment, can be optimal. The second moment is single optimal by (Theorem 2)

$$\xi \in \left[\frac{1}{6}; \frac{4}{9} \right), \tag{107}$$

the third moment is single optimal by (52) (Theorem 3), and the final moment is single optimal by (68) (Theorem 6), whereas the third and final moments in the 4×4 duel are both optimal at the boundary value of jitter (53) (Theorem 3).

The 5×5 duel is pure strategy solvable by any possible jitter

$$\xi \in \left\{ \left(-\frac{1}{3}; \frac{8}{27} \right) \setminus \left\{ \left(\frac{\sqrt{19}-4}{9}; \frac{1}{6} \right] \cup \left(-\frac{11}{54}; -\frac{1}{18} \right) \right\} \right\}. \tag{108}$$

The 6×6 duel is pure strategy solvable by any possible jitter

$$\xi \in \left\{ \left(-\frac{1}{3}; \frac{16}{81} \right) \setminus \left\{ \left(\frac{\sqrt{19}-4}{9}; \frac{1}{6} \right] \cup \left(-\frac{49}{162}; -\frac{1}{18} \right) \right\} \right\}. \tag{109}$$

In 5×5 and 6×6 duels only the second (Theorem 2), third (Theorem 3), and final (Theorem 7) moments can be optimal.

Duels with seven to nine time moments are pure strategy solvable only by (50). Bigger duels, having no fewer than 10 time moments, are pure strategy solvable only by (51). Unlike duels with the geometrical progression pattern, the third time moment is the only possible pure strategy solution in duels having no fewer than seven time moments (Theorems 3 and 7). However, the jitter interval, at which the pure strategy solution exists, gradually vanishes as the duel size increases off 10. In 7×7, 8×8, and 9×9 duels, to the contrary, the jitter interval, at which the pure strategy solution exists, is constant.

Overall, as the duel becomes bigger, its pure strategy solvability worsens — the subset of jitter values, at which pure strategy solutions exist, becomes narrower (except for duels with seven to nine time moments). Another distinct property is that the optimal time moment, if any, moves away from the duel beginning as jitter becomes lower (the positive jitter becomes lower or the negative jitter becomes lower).

The duel pure strategy solutions obtained above suggest a clear one-step-action strategic behaviour in progressive block proposal timing for decentralized consensus protocols under uncertainty of time slots to act [1], [3], [17], [18]. The main benefit is the full fairness and a potential reward if the opponent acts non-optimally, even in a single proposal [19], [20]. When the network receives two blocks at once, the network splits briefly: some nodes build on the first duelist's block, others on the second duelist's. This temporarily creates two competing chains (a network fork) at the same height [2]. Nevertheless, most Proof-of-Stake blockchains, for instance, have rules that resolve forks deterministically by the criteria of which block was seen first by the majority, which validator has higher stake weight or reputation, which block has more valid attestations (votes from other validators) [4], [6]. As a result, one block wins, and the other one is orphaned: one block becomes part of the canonical chain, the other one is discarded [1], [5]. Further research may be focused on considering other progression patterns and a decaying value of loss.

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Романюк В.В.

Оптимальні стратегії часу у пропозиціях блокчейн-блоків засобами однокульових безшумних дуелей з однотретинною прогресією

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Проблематика. Безшумні дуелі та пов'язані з ними ігри на вибір часу пропонують несподівано глибоке розуміння певних ключових викликів у технології блокчейну, особливо щодо таймінгу пропозиції блоку. Майнери або валідатори фактично "змагаються" у гонці за право запропонувати наступний блок. Успіх пропозиції блоку залежить не тільки від моменту його здійснення, а й від того, чи вже хтось інший не досяг успіху або не завадив цьому процесу — дуже схоже на напругу у дуелі з одним пострілом і невизначеним результатом. У таймінгу пропозиції блоку для децентралізованих протоколів консенсусу одноразова гра на вибір часу моделює ситуацію в блокчейні, де учасники (наприклад, валідатори або майнери) обирають момент для спроби запропонувати блок або вставити транзакцію за умов невизначеності.

Мета дослідження. Загальною метою є визначення найкращих стратегій таймінгу для учасників. Розглядаючи двох ідентичних учасників, локальна мета полягає у знаходженні розв'язків у чистих стратегіях гри на вибір часу (дуелі) за рівномірного джитера моменту пострілу.

Методика реалізації. Розглядається скінченна гра з нульовою сумою, яка моделює конкурентну взаємодію між двома суб'єктами у прийнятті найкращого дискретного рішення за умов обмеженої спостережуваності. Моменти для прийняття рішення (виконання дії, пострілу кулею) призначаються заздалегідь, і кожен із суб'єктів, яких також називають дуелянтами, має лише одну кулю для пострілу. Стріляти дозволено тільки протягом стандартизованого проміжку часу, де кулю можна випустити лише у визначені моменти часу. У базовій моделі, окрім початкового та кінцевого моментів дуелі, кожен наступний момент визначається додаванням третини залишкового проміжку до поточного моменту. Однак точна специфікація моментів часу не завжди здійсненна (наприклад, через обмежену точність вимірювання відстані між сусідніми моментами), тому внутрішні моменти часу зазнають рівномірного джитера. Це означає, що вони можуть бути злегка зміщені в межах проміжку дуелі. Дуелянту вигідно стріляти якомога пізніше, але тільки якщо він стріляє першим. Обидва дуелянти діють в однакових умовах за лінійної влучності пострілу, тому дуель із однією кулею є симетричною, незалежно від джитера. Відповідно, її оптимальне значення дорівнює 0, а дуелянти мають однакові оптимальні стратегії, хоча ці стратегії можуть бути також і несиметричними.

Результати дослідження. За моделлю однотретинної прогресії з джитером 3×3-дуель завжди має розв'язок у чистих стратегіях. 4×4-Дуель розв'язується у чистих стратегіях за будь-якого можливого джитера, окрім інтервалу джитера $\left(\left(\sqrt{19}-4\right)/9;1/6\right)$. У межах цього інтервалу та інтервалу $(-11/54;-1/18)$ 5×5-дуель не розв'язується у чистих стратегіях. 6×6-Дуель розв'язується у чистих стратегіях за будь-якого можливого джитера, окрім інтервалів $\left(\left(\sqrt{19}-4\right)/9;1/6\right)$ та $(-49/162;-1/18)$. Дуелі із семи до дев'яти моментів часу розв'язуються у чистих стратегіях лише при джитері в інтервалі $\left[-1/18;\left(\sqrt{19}-4\right)/9\right]$. Більші $N\times N$-дуелі, де кількість моментів часу не менше 10, розв'язуються у чистих стратегіях тільки при джитері в інтервалі $\left[-1/18;2^{N-2}/3^{N-2}\right)$. Розв'язки для моделі однотретинної прогресії порівнюються з відомими розв'язками для моделі геометричної прогресії.

Висновки. Отримані розв'язки у чистих стратегіях для дуелей пропонують чітку однокрокову поведінку при прогресуючому таймінгу пропозицій блоків у децентралізованих протоколах консенсусу за умов невизначеності часових слотів для дій. Основною перевагою є повна справедливість і можливість отримати винагороду, якщо опонент діє неоптимально, навіть за однієї спроби пропозиції.

Ключові слова: *таймінг пропозицій блоків; безшумна дуель з однією кулею; лінійна влучність; матрична гра; розв'язок у чистих стратегіях; прогресуючі на третину моменти пострілу.*

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