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ASYMPTOTIC PROPERTIES OF SELF-SIMILAR TRAFFIC MODELS BASED ON DISCRETE-TIME AND CONTINUOUS-TIME MARTINGALES

A. Pilipenko, Leonid O. Uryvskyi, B. Trach

NTUU "KPI", Institute of telecommunication system, Kyiv, Ukraine

Asymptotic properties of self-similar traffic models based on discrete-time and continuous-time martingales are considered. We discovered that their performance indicators are asymptotically equal at $\lambda \to \infty$ to indicators for model based on Brownian motion.

Introduction

Massive implementation of computer networks and increases in their productivity turned attention of scientists studying network traffic to the properties of the Internet traffic. In 1993, Willinger et al. [1] discovered that computer network traffic exhibits statistical self-similarity. This phenomenon can be described by three features — non-integer fractal dimension (hence, the object is fractal), scale invariance — statistical parameters are independent of the level of flow aggregation and time scale, and long-range dependence. Autocorrelation function of process with long-range dependence decays slower than exponential function. This property is not present in telephone network traffic — it possesses only scale invariance. Therefore, new approaches to network parameter estimation should be developed.

The numerical parameter of self-similarity is the Hurst exponent [2], which lies between 0 and 1. When the Hurst exponent is between 0 and 0.5, the stochastic process is antipersistent. In this case, stochastic process exhibits no trends. When the Hurst exponent is greater than 0.5, it exhibits persistence, which means that long trends are formed in the process. It was estimated that the Hurst exponent for network traffic approximately equals 0.8. It is possible to accurately describe traffic structure using this parameter.

Numerous stochastic processes are utilized for research and development of network traffic models. Martingales are one of the examples of such process. Martingale is a discrete or continuous process which satisfies the following condition

$$M\left[X(t) \mid \{X(\tau), \tau < s\}\right] = X(s), \forall s \le t .$$

One can derive the following property from this equation: the expectation of process is constant throughout the realization of the process. Wide range of stochastic processes exhibits this property; for example, Brownian motion is a martingale. Martingales are commonly used for analysis of time series.

Problem statement

The purpose of the research is the study of asymptotic properties of discrete and continuous models of network traffic with discrete and continuous time. The object of the research is asymptotic properties of network traffic models. The subject of the study is application of martingales for network traffic modeling.

Research of asymptotic properties

To describe our approach, let's at first consider the M/M/1/m system with arrival intensity λ and processing intensity μ.

The waiting requests for processing get into a queue with the buffer capacity of m requests and get lost at the buffer overflow.

Let $X_{\lambda,\mu}(t)$ be a number of requests in a system at time t; $N_\lambda(t)$ be a number of requests that arrived before t; $N_\mu(t)$ be a number of requests that could be processed up to time t if there are no idle time. It is well-known [3] that $N_\lambda(t)$ and $N_\mu(t)$ are independent Poisson processes with intensities λ and μ, respectively.

If there were neither losses during period $[0, t]$, nor idle times, then

$$X_{\lambda,\mu}(t) = X_{\lambda,\mu}(0) + N_\lambda(t) - N_\mu(t) \qquad (1)$$

A number of requests in the system cannot be less than 0 physically: $X_{\lambda,\mu}(t) \ge 0$. During the period, when $X_{\lambda\mu}(0)+N_\lambda(t)-N_\mu(t) < 0$ the system is non-busy. Calculated below of the threshold "0" requests should be considered as virtually processed, i.e. actually not served.

Therefore, in general case we have the following representation:

$$X_{\lambda,\mu}(t) = X_{\lambda,\mu}(0) + N_\lambda(t) - \\ - N_\mu(t) + L_0^{\lambda,\mu}(t) - L_m^{\lambda,\mu}(t),$$

where $L_m(t)$ is the number of lost requests (above the threshold m), $L_0(t)$ is the number of virtually processed requests (below the threshold 0).

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Observe that processes X, L_0, L_m can be considered as a solution of two-sided Skorokhod's problem. Let us recall the corresponding definition.

Definition [4]. Let f be a function, $f(0) \in [0, m]$. We say that functions g, L_0, L_m satisfy two-sided Skorokhod's reflection problem with reflections at 0 and m if

1) $g(t) = f(t) + L_0(t) - L_m(t), t \geq 0;$
2) $L_0(0) = L_m(t) = 0$, L_0 and L_m are non-decreasing;
3) L_0 and L_m may increase only when g equals 0 or m, respectively, i.e.

$$\int_0^t 1_{g(s)>0} dL_0(s) + \int_0^t 1_{g(s)<m} dL_m(s) = 0,\ t \geq 0,$$

4) $g(t) \in [0, m], t \geq 0.$

It is known [5] that there is a unique solution of the Skorokhod's problem (functions g, L_0, L_m are unknown) for a given function f.

This solution in some sense depends continuously on f.

In our model:

$$f(t) = X_{\lambda,\mu}(0) + N_\lambda(t) - N_\mu(t),$$

$$g(t) = X_{\lambda,\mu}(t)$$

Let us assume that λ, μ, m are large enough in such a way that

$$\mu = \lambda + a\sqrt{\lambda} + o(\sqrt{\lambda}),$$
$$m = \beta\sqrt{\lambda} + o(\sqrt{\lambda}), \qquad (2)$$
$$X_{\lambda,\mu}(0) = x\sqrt{\lambda} + o(\sqrt{\lambda}), \lambda \to +\infty$$

By invariance principle [6] a process $\frac{N_\lambda(t) - \lambda t}{\sqrt{\lambda}}$ converges weakly to Brownian motion $B(t)$ as $\lambda \to +\infty$. So

$$X_{\lambda,\mu}(t) = X_{\lambda,\mu}(0) + \left(\lambda t + B(t)\sqrt{\lambda}\right) - \left(\left(\lambda + a\sqrt{\lambda}\right)t + \tilde{B}(t)\sqrt{\mu}\right) + \varepsilon_{\lambda,\mu}(t) + L_0^{\lambda,\mu}(t) - L_m^{\lambda,\mu}(t), \qquad (3)$$

where $\tilde{B}(t)$ is a Brownian motion independent on $B(t)$, $\frac{\varepsilon_{\lambda,\mu}(t)}{\sqrt{\lambda}} \to 0$ as $\lambda \to \infty$.

It follows from (3) that a process $Y_{\lambda,\mu}(t) = \frac{X_{\lambda,\mu}(t)}{\sqrt{\lambda}}$ converges as $\lambda \to \infty$ to a solution of the following Skorokhod's problem on $[0, \beta]$:

$$Y(t) = x + \sqrt{2}w(t) - at + \widetilde{L_0}(t) - \widetilde{L_\beta}(t),$$

where $w(t)$ is a Brownian motion.

Remark. A process $B(t) - \tilde{B}(t)$ has the same distribution as $\sqrt{2}w(t)$. Thus,

$$X_{\lambda,\mu}(t) \approx \sqrt{\lambda}Y(t);\ L_m^{\lambda,\mu}(t) \approx \sqrt{\lambda}\widetilde{L_\beta}(t) \qquad (4)$$

Note that it is very easy to find a rejection probability for $X_{\lambda,\mu}(t)$ [3]:

$$\pi_m = \frac{(\lambda/\mu)^m}{1 + (\lambda/\mu) + \ldots + (\lambda/\mu)^m} = \frac{(\lambda/\mu)^m(1 - \lambda/\mu)}{1 - (\lambda/\mu)^{m+1}}$$

It follows from (2) that

$$\pi_m \sim \frac{ae^{-a\beta}}{1 - e^{-a\beta}}\frac{1}{\sqrt{\lambda}} = \frac{a}{e^{a\beta} - 1}\frac{1}{\sqrt{\lambda}}, \lambda \to \infty \qquad (5)$$

If we consider more complicate model than M/M/1/m, then it could be very difficult to find the explicit formula for rejection probability. However, the invariance principle holds true under very insignificant assumptions. For example, there is no need to assume the exponential distribution between arrivals (or processing) as it is for a Poisson process. It is sufficient to suppose that these times are i.i.d. with finite expectation. We may also assume that each arrival time a group of requests arrive (the second moment of a group should be finite). The collective processing is permitted too. In any case a limit will be of the form

$$Y(t) = x + bw(t) - at + \widetilde{L_0}(t) - \widetilde{L_\beta}(t) \qquad (6)$$

Remark. For the M/M/1/m model a constant b equals $\sqrt{2}$, but for a general model, (say for a group arrivals) a constant b can be arbitrary.

It is much easier to investigate a simple continuous model (6), then a very general discrete model. The rejection probability for a discrete model equals $\lim_{t\to\infty} \frac{L_m^{\lambda,\mu}(t)}{t}$. This approximately equals $\lim_{t\to\infty} \frac{\widetilde{L_\beta}(t)}{\sqrt{\lambda}t}$. So, let's calculate this limit for a general model (6). It can be proved [7] that

$$\lim_{t\to\infty} \frac{L_\beta(t)}{t} = \lim_{t\to\infty} \frac{EL_\beta(t)}{t} = \frac{b^2}{2}\pi(\beta),$$

where $\pi(y), y \in [0, \beta)$ is a stationary density for a process $Y(t)$. It can be found from the Fokker-Planck-Kolmogorov equation [8] for $Y(t)$:

$$\frac{1}{2}b^2\pi''(y) + a\pi'(y) = 0.$$

So,

$$\pi(y) = \frac{\frac{2a}{b^2}e^{-\frac{2ay}{b^2}}}{1 - e^{-\frac{2a\beta}{b^2}}}, y \in [0, \beta]; \qquad (7)$$

and

$$\lim_{t\to\infty} \frac{\widetilde{L_\beta}(t)}{t} = \frac{\frac{b^2 2a}{2\ b^2}e^{-\frac{2a\beta}{b^2}}}{1 - e^{-\frac{2a\beta}{b^2}}} = \frac{a}{e^{\frac{2a\beta}{b^2}} - 1}. \qquad (8)$$

This completely agrees with (5), (4), when $b = \sqrt{2}$.

It can be seen that that rejection probabilities for extreme case of discrete Markov process (5) and continuous Markov process (7) are the same. Similar results can be obtained for other characteristics of the processes.

It can be inferred that in extreme case of discrete and continuous processes all output parameters of the model are described by the same equations. As a result, simpler continuous model can be used to calculate parameters of network traffic in queuing systems with complex structure or traffic with priorities.

Dependence between rejection probability and the main parameters has been investigated. The main parameters are a — coupling coefficient between λ and μ, and b — buffer scaling factor. As it is assumed that $\lambda \to \infty$, changing λ won't have any noticeable effect on the process.

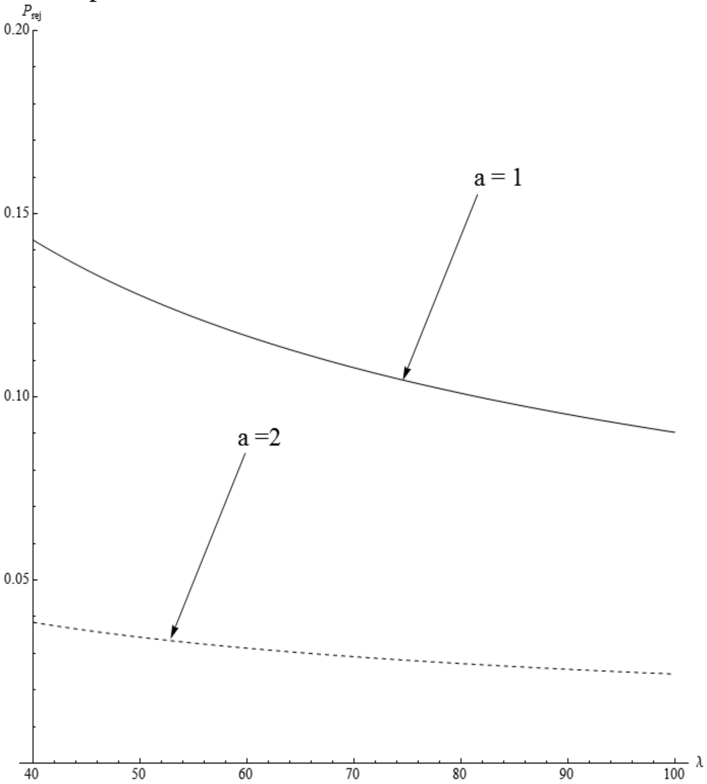

Fig. 1. Graphical dependence of rejection probability on the value of λ for different values of a: $a=1$ and $a=2$, with parameters $b=1, m=1$.

The dependence of rejection probability on the arrival intensity λ for different values of coupling factor a is shown on Figure 1. When the service probability increases, the rejection probability decreases. This behavior is consistent with physical sense of processes in the buffer and results of modeling for continuous and discreet Markov processes.

Conclusions

Our study of discrete and continuous Markov processes has shown that in extreme case of arrival intensity $\lambda \to \infty$ and values of service intensity, which are of the same order of magnitude, the output parameters of the model (service, rejection probabilities, etc.) are described by the same equations.

The practical result of this work is the fact that a lot less complicated continuous queuing system model based on the Brownian motion can be used to calculate parameters of network traffic of complex queuing systems. For example, in order to find state probabilities for queuing system featuring traffic with priorities using Poisson model, a large system of linear equations must be solved. Brownian model gives us ability to find these probabilities by solving one equation.

We have conducted the research for rejection probability. But other output parameters of performance and latency can be obtained in similar fashion. Also, more complex queuing systems can be considered (for example, with multiple servers). In addition to that, further research must be conducted with the use of fractional Brownian motion, which would give us ability to study properties of processes with long-range dependence. All of this will give us practical models of communication networks [9].

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